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FOUNDATIONS, THEORY OF SETS, LOGIC

Geach, P. T. On Frege's way out. Mind **65** (1956), 408–409.

Shoenfield, J. R. Quasicreative sets. Proc. Amer. Math. Soc. **8** (1957), 964-967.

This paper deals with collections of non-negative integers, henceforth called sets. Let $\Phi_n(z, x_1, \cdots, x_n)$ be the partial recursive function of $n+1$ arguments defined on p. 340 of S. C. Kleene's "Introduction to metamathematics" [Van Nostrand, New York, 1952; MR **14**, 525]. For every n we denote the range of $\Phi_1(n, x)$ by ω_n. All recursively enumerable (r.e.) sets are now effectively generated in the sequence $\{\omega_n\}$. Let all non-empty finite sets be effectively generated without repetitions in the sequence $\{\alpha_n\}$ such that the cardinality of α_n is a recursive function of n. The domain of a function $p(n)$ is denoted by δp. A set σ is productive if there is a partial recursive function $p(n)$ such that $\omega_n \subset \sigma$ implies:

$$n \in \delta p \ \& \ p(n) \in \sigma - \omega_n.$$

A creative set is a r.e. set with a productive complement. Creative sets were introduced by E. L. Post [Bull. Amer. Math. Soc. **50** (1944), 284–316; MR **6**, 29]; see also J. Myhill [Z. Math. Logik Grundlagen Math. **1** (1955), 97–108; MR **17**, 118].

The author calls a set σ quasiproductive if there is a partial recursive function $p(n)$ such that $\omega_n \subset \sigma$ implies:

$$n \in \delta p \ \& \ \alpha_{p(n)} \subset \sigma \ \& \ \alpha_{p(n)} \not\subset \omega_n.$$

A quasicreative set is a r.e. set with a quasiproductive complement. Clearly, every productive set is quasiproductive and every creative set is quasicreative. Theorems: A set σ is quasiproductive if and only if

$$\alpha_{f(n)} \subset \sigma \Leftrightarrow \alpha_{f(n)} \not\subset \omega_n$$

for some recursive function $f(n)$. Every r.e. set is reducible by truth tables to every quasiproductive set. There exist disjoint r.e. sets α and β such that α is quasicreative and $\alpha+\beta$ simple. If α and β are disjoint r.e. sets whose union is simple, no creative set is reducible to α by bounded truth tables. Corollaries: There exist quasicreative sets which are not creative. All quasicreative sets are reducible to each other by truth tables, but not by bounded truth tables. There exist two r.e. sets which are inseparable by recursive sets, but not effectively inseparable. The author points out that the last statement has previously been proved by A. A. Muchnik [Dokl. Akad. Nauk SSSR (N.S.) **109** (1956), 29–32; MR **18**, 866]. *J. C. E. Dekker.*

Grzegorczyk, A. On the definitions of computable real continuous functions. Fund. Math. **44** (1957), 61–71.

The author's original definition (I) of computable real continuous function was proposed in Fund. Math. **42** (1955), 168–202 [MR **19**, 238], under the name "computable real function". It was based on the notion of computable functionals. In the present paper, four alternative definitions (II, II′, III and III″) based on computable sequences of real numbers, and two (IV and IV′) based on computable sequences of open rational segments, are proved equivalent to I.

Under II, a real function φ is computable continuous if and only if (i) for each computable sequence $\{a_n\}$ of reals, the sequence $\{\varphi(a_n)\}$ is computable and (ii) φ is computably uniformly continuous (in an obvious sense) with respect to the rational segments. For II′, (i) is replaced by the condition that, for a recursive enumeration of the rationals r_n without repetition, the sequence $\{\varphi(r_n)\}$ is computable. For III and III′, φ is required to be continuous, so that (ii) can be formulated in terms of rational arguments.

Definitions IV and IV′ are similar to that of D. Lacombe [C. R. Acad. Sci. Paris **240** (1955), 2478–2480; **241** (1955), 13–14, 151–153; MR **17**, 225]. *G. F. Rose.*

★ **Nidditch, P. H. Introductory formal logic of mathematics.** University Tutorial Press Ltd., London, 1957. vii + 188 pp. 12s. 6d.

The book is written both as a textbook for second year logic students and as the first of a series of volumes for mathematicians. In the first third of the book a logic of natural deduction for the propositional and predicate logics is given, and in the latter two thirds this logic is applied in the development of a set theory and the proof of a number of elementary theorems in the theory of relations. A bracketless notation for the propositional and predicate logics, combined with the author's own notation for set theory, is used.

The author claims that the book "initiates a radically new departure in Mathematics since, for the first time, it gives logically valid proofs of mathematical theorems..." The claim is false on historical and technical grounds. Historically the "new departure" was initiated by Frege and has been continued by a number of mathematicians. Technically, a precise definition of "logically valid proof" is essential to the author's aim; and yet, despite the author's demands of rigor from mathematicians, the book falls far short of generally accepted standards of precision in the exposition of formal logic. The statements of some of the rules of deduction, for example, are unsatisfactory since they contain either undefined or incompletely defined terms.

It is unfortunate that, in a series of inexpensive university texts likely to be widely read by interested students, logic is represented by this book. Oversimplifi-

cations regarding the work of contemporary mathematicians, exaggerations regarding the importance of this book, the use of an unconventional notation and sins of omission and commission (for example, § 25, p. 86: every set of ordered pairs is said to be the cartesian product of two sets) make this book unsuitable for students.

P. C. Gilmore (Ossining, N.Y.).

Kemeny, John G. A contribution to inductive logic. Philos. and Phenomenol. Res. **13** (1953), 371–374; discussion, 375–376.

Łoś, Jerzy. On logical matrices. Trav. Soc. Sci. Lett. Wrocław. Ser. B. no. **19** (1949), 42 pp. (Polish)

Łoś, J.; and Suszko, R. On the extending of models. IV. Fund. Math. **44** (1957), 52–60.

For parts I–III see Łoś, Fund. Math. **42** (1955), 38–54; Łoś and Suszko, ibid. **42** (1955), 343–347; Słominski, ibid. **43** (1956), 69–76; MR **17**, 224, 815; **18**, 2.] Let $\{\mathfrak{M}_n\}$ be an increasing sequence of models of a first order theory E_1 with identity. A sentence α of E_1 is called persistent if, for every n,

$$\alpha \in E_1(\mathfrak{M}_n) \rightarrow \alpha \in E_1\left(\sum_{n=1}^{n=\infty} \mathfrak{M}_n\right).$$

The authors show that just those sentences are persistent in every increasing sequence of models which are (equivalent to) $\prod \sum$ sentences. [Cf. S. Robinson, J. Symb. Logic **21** (1956), 33–35; MR **17**, 817.] *G. Kreisel.*

Łoś, J. Remarks on Henkin's paper: Boolean representation through propositional calculus. Fund. Math. **44** (1957), 82–83.

In his first remark the author gives a short proof of Henkin's theorem that the Boolean representation theorem follows from the completeness of the propositional calculus in the sense of Gödel [Ergebn. Math. Kolloq. **3** (1930/31), 20–21] and Malcev [Mat. Sb. N.S. **1(43)** (1936), 323–336]. Also, he clarifies a passage in Henkin's sketchy proof of the converse. In his second remark he shows, also without the use of the axiom of choice, that the G.-M. theorem implies: If U is a class of ideals of a Boolean algebra, then there is a U' of equal power to U such that every I (in U)$\subset I'$, $I' \in U'$ and I' prime. *G. Kreisel.*

Montague, Richard; and Kalish, Donald. Remarks on descriptions and natural deduction. Arch. Math. Logik Grundlagenforsch. **3** (1957), 50–64.

In these first 15 pages of their paper the authors set up a system for the predicate calculus with descriptions and establish its completeness. Improper descriptions are allowed as terms, and are interpreted to denote the 'null-entity'. In the last section a perspicuous comparison is made with other systems of description, in particular by Carnap [Meaning and necessity, Univ. of Chicago Press, 1947; MR **8**, 430], Hailperin [J. Symb. Logic **19** (1954), 14–20; MR **15**, 845], Rosser [Logic for mathematicians, McGraw-Hill, New York, 1953; MR **14**, 935] and in the classical works of Whitehead-Russell and Hilbert-Bernays. Also some misstatements in the literature are corrected. The paper by Schröter [Z. Math. Logik Grundlagen Math. **2** (1956), 37–56; MR **17**, 1173] is not examined.

G. Kreisel (Reading).

Mostowski, A. On a generalization of quantifiers. Fund. Math. **44** (1957), 12–36.

The author's generalization of the existential and universal quantifier $[(Ex)$ and $(x)]$ is this: Let the predicate $T_{\mathfrak{m}}(\mathfrak{n}, \mathfrak{p})$ be defined for every cardinal $\mathfrak{m}$, and all $\mathfrak{n}$, $\mathfrak{p}$ satisfying $\mathfrak{n}+\mathfrak{p}=\mathfrak{m}$. $(Q_T x)F(x)$ is true in a domain of cardinality $\mathfrak{m}$ if and only if $F(x)$ is true for just $\mathfrak{n}$ elements of D and false for $\mathfrak{p}$ elements. Thus (Ex) is defined by T with $T_{\mathfrak{m}}(\mathfrak{n}, \mathfrak{p})$ true except for $\mathfrak{n}=0$, (x) by T' with $T_{\mathfrak{m}}'(\mathfrak{n}, \mathfrak{p})$ true for $\mathfrak{p}=0$, and the numerical quantifiers $(E^{(n)}x)$ $[(x)^{(n)}]$ by T with $T_{\mathfrak{m}}(\mathfrak{n}, \mathfrak{p})$ true just for $\mathfrak{n}=n$ $[\mathfrak{p}=n]$. $P(Q^1, \cdots, Q^n; F_1, \cdots, F_p)$ denotes the class of formulae built up analogously to the classical predicate calculus with Q^i, $1 \leqq i \leqq n$, instead of (Ex) and (x) and the non-logical constants F_j, $1 \leqq j \leqq p$. The theory of $P(Q^1, \cdots, Q^n, F_1, \cdots, F_p)$ is said to have an $(I$-) complete formalization if the set of formulae valid in all domains (of cardinality I) is recursively enumerable. — The author shows that the usual predicate calculus permits essentially the only $\aleph_0$-complete formalization. — Th. 4: If (Ex) and (x) occur among $Q^1, \cdots, Q^n$, and if $P(Q^1, \cdots, Q^n, F_1, \cdots, F_p)$ is I-complete, then each Q^i is a numerical quantifier. Also Th. 8: If (Ex) and (x) occur among $Q^1, \cdots, Q^n$ and if every formula of $P(Q^1, \cdots, Q^n, F_1, \cdots, F_p)$ which is satisfiable in one infinite domain is satisfiable in all infinite domains, then each Q^i is definable by means of Boolean operations in terms of (Ex) and (x). This too isolates a characteristic property of the classical predicate calculus. The paper concludes with an analysis of the decision method for the monadic predicate calculus from the present point of view, when F_i^1 are monadic predicate symbols. In the decision method every formula of $P(Q^1, \cdots, Q^n, F_1^1, \cdots, F_p^1)$ is reduced to a Boolean combination of formulae $(Q^j x)[\cdots \& G_i(x) \& \cdots]$, $1 \leqq i \leqq p$, where $G_i(x)$ is either $F_i^1(x)$ or $\neg F_i^1(x)$, provided the essential step in the usual elimination $\{(Ex)[A(x) \vee B(x)] \leftrightarrow [(Ex)A(x) \vee (Ex)B(x)]\}$ can be generalized to $(Q^j x)[A(x) \vee B(x)] \leftrightarrow [(Q^j x)A(x) \vee (Q^j x)B(x)]$. In this case it follows (Theorem 11) that the class of valid sentences of $P(Q^1, \cdots, Q^n, F_1^1, \cdots, F_p^1)$ is recursive, although with particular quantifiers one may not be able to decide whether $(Q^j x)[\cdots \& G_i(x) \& \cdots]$ is satisfiable. This can be done (Theorem 10) for sets of quantifiers Q^j satisfying the following conditions: $(Q^j x)F_i^1(x)$ if and only if $F_i^1(x)$ is true for $\geqq m_j$ elements and $m_1 < m_2 < \cdots < m_n$, $m_2, \cdots, m_n$ infinite, and m_1 either $=1$ or infinite.

G. Kreisel.

Schmidt, H. Arnold. Das fundamentale Implikationensystem einer implikativen Modalitätenstruktur mit idempotenter Möglichkeit. Arch. Math. Logik Grundlagenforsch. **2** (1956), 33–54.

This paper investigates the implications which hold among logical modalities. It is assumed here that the basic modalities 'possible' and 'necessary' are idempotent. The notion of an implicative modal structure is defined by a set of axioms each of which is an implication of the form $\alpha \rightarrow \beta$, where α and β are modalities. It turns out that there is a finite number of such idempotent implicative modal structures. The present paper contains the basic definitions and theorems which are necessary to describe all the idempotent implicative modal structures. Their exact description will follow in a later paper. *L. N. Gál.*

See also: General Topology: Papić.

ALGEBRA

Nolin, Louis. Sur les classes d'algèbres équationnelles et les théorèmes de représentation. C. R. Acad. Sci. Paris **244** (1957), 1862–1863.

Il existe un procédé général de démonstration des théorèmes de représentation pour certaines classes d'algèbres; appliqué aux algèbres de Boole, il permet d'obtenir le théorème de Stone de manière élémentaire.

Résumé de l'auteur.

Linear Algebra

★ **Halmos, Paul R. Finite-dimensional vector spaces.** 2nd ed. The University Series in Undergraduate Mathematics. D. Van Nostrand Co., Inc., Princeton-Toronto-New York-London, 1958. viii+200 pp.

The first edition was reviewed in MR **4**, 11. Apart from minor changes, the present edition contains the following new material: discussion of fields, definition of determinants in invariant terms, and over 300 exercises.

★ **Проскуряков, И. В. [Proskuryakov, I. V.] Сборник задач по линейной алгебре. [Collection of exercises in linear algebra.]** Gosudarstv. Izdat. Tehn.-Teor. Lit., Moscow, 1957. 368 pp. 7.50 rubles.

The book contains 1753 exercises under the headings: determinants, systems of linear equations, matrices and quadratic forms, vector spaces and their linear transformations; with an appendix on groups, rings and fields. Answers and, in some cases, a discussion of methods of solution are given.

de Fátima Fontes de Sousa, Maria. Some observations on invariant subspaces of a matrix. Univ. Lisboa. Revista Fac. Ci. A. (2) **5** (1956), 353-360. (Portuguese)

L'auteur étudie les espaces invariants d'une matrice et la réduction canonique de celle-ci. *G. Papy* (Bruxelles).

★ **Browne, Edward Tankard. Introduction to the theory of determinants and matrices.** University of North Carolina Press, Chapel Hill, N. C., 1958. xi+270 pp. $7.50.

A text intended for seniors and first year graduate students. There are 20 chapters with the following topics: fundamental concepts, elementary properties of determinants, elementary transformations on a matrix, algebra of matrices, linear dependence, linear equations, characteristic equation of a matrix, special types of matrices, bilinear forms, quadratic forms, real quadratic forms, lambda-matrices, equivalence of pairs of matrices, reduced characteristic function of a matrix, canonical forms of a matrix, scalar polynomials in a matrix, reduction of a matrix to a canonical form, equivalence of pairs of forms, commutative matrices, systems of differential equations.

★ **Valentiner, Siegfried. Vektoren und Matrizen.** 8., erweiterte Aufl. der "Vektoranalysis". Mit einem Anhang: Aufgaben zur Vektorrechnung von Dr. Hermann König. Sammlung Göschen Bd. 354/354a. Walter de Gruyter & Co., Berlin, 1958. 202 pp. DM 4.80.

The 7th edition was listed in MR **15**, 609.

Brauer, Alfred. A new proof of theorems of Perron and Frobenius on non-negative matrices. I. Positive matrices. Duke Math. J. **24** (1957), 367-378.

The main aim of this paper is to give new proofs for Frobenius' and Perron's results concerning positive matrices, namely that the dominant characteristic root is positive and simple and that the corresponding vector can be chosen positive. These facts are derived by continuity arguments from a special class of positive matrices, namely stochastic matrices which have constant positive row sums and which then have, trivially, a positive dominant characteristic root. First, a chain of similarity transformations via positive diagonal matrices is applied to obtain a matrix for which the maximum and minimum row sums are arbitrarily close. Using this, a stochastic matrix is constructed whose roots (in a certain order) are arbitrarily close to the roots of the given matrix. At the same time, it can also be shown that a stochastic matrix exists whose elements are arbitrarily close to a matrix which is similar to the given matrix. (Using the positiveness of the dominant root, the existence of a similarity transformation between a positive and a stochastic matrix was noticed by Kolmogoroff [see N. Dimitriev and E. Dynkin, Izv. Akad. Nauk SSSR. Ser. Mat. **10** (1946), 167–184; MR **8**, 129]). The method leads to new bounds for the dominant root of a positive matrix and of its $n-1\times n-1$ principal minors, using the bounds for stochastic matrices previously obtained by the author [Duke Math. J. **19** (1952), 75–91; MR **13**, 813]. The same idea furnishes a Cassini oval which encloses all the characteristic root. *O. Taussky-Todd* (Pasadena, Calif.).

Taussky, Olga. A determinantal inequality of H. P. Robertson. I. J. Washington Acad. Sci. **47** (1957), 263–264.

Let $H=A+iB$ be an n-square Hermitian matrix with A and B real. Robertson [Phys. Rev. (2) **46** (1934), 794–801] showed that if H is positive definite (p.d.), then (1) $\det A>\det B$. The author shows that Robertson's proof yields the stronger result: H is p.d. if and only if the eigenvalues of $iA^{-1}B$ are real and <1. Using the same method of proof she shows that if H is p.d., then (2) $\det H\leqq\det A$, with equality if and only if $B=0$. In fact, if $E_r(M)$ is the rth elementary symmetric function of the eigenvalues of the matrix M, then $E_r(H)\leqq E_r(A)$ for $r=2, \cdots, n$. The following generalization of (1) is also proved. Let $\alpha_1\geqq\cdots\geqq\alpha_n$ and $\beta_1\geqq\cdots\geqq\beta_n$ be the eigenvalues of A and iB, respectively; if H is p.d., then $\alpha_i>\beta_i$ $(i=1, 2, \cdots, n)$. *B. N. Moyls.*

Marcus, Marvin. A determinantal inequality of H. P. Robertson. II. J. Washington Acad. Sci. **47** (1957), 264–266.

We shall use the notation of the preceding review. Let H and K be positive definite n-square Hermitian matrices. The author shows that $\det^{1/n}(H+K)^2\geqq\det^{1/n}(H-K)^2+4\det^{1/n}(HK)$, with equality if and only if $K^{-1}H+H^{-1}K=cI$ for a constant $c\geqq 2$. It follows, when $H=A+iB$ and $K=\bar{H}$, that $\det A\geqq|\det B|+\det H$ [cf. (1) and (2) of the preceding review]. If $1\leqq r\leqq n$, then $E_r(iB)=0$ when r is odd, and $(-1)^s E_r(iB)\leqq E_r(A)-E_r(H)$ when $r=2s$. Let the eigenvalues of H, A, and iB be $h_1\geqq\cdots\geqq h_n$, $\alpha_1\geqq\cdots\geqq\alpha_n$, and $\beta_1\geqq\cdots\geqq\beta_n$, respectively; and let $1\leqq k\leqq n$. Then

(a) $$\prod_{j=1}^{k}\alpha_{n-j+1}\geqq\prod_{j=1}^{k}h_{n-j+1};$$

if B is non-singular,

(b) $$\prod_{j=1}^{k} \alpha_j^{2/k} \geqq \prod_{j=1}^{s} \beta_j^{4/k} + \prod_{j=1}^{k} h_{n-j+1}^{2/k}$$

when $k=2s$, and

(c) $$\prod_{j=1}^{k} \alpha_j^{2/k} \geqq \beta_{s+1}^{2/k} \prod_{j=1}^{s} \beta_j^{4/k} + \prod_{j=1}^{k} h_{n-j+1}^{2/k}$$

when $k=2s+1$. ((c) is stated incorrectly in the paper.) It follows from (a) that if f is such a function that $f(e^{t_1}, \cdots, e^{t_k})$ is increasing in each t_j, convex and symmetric, then $f(h_1, \cdots, h_k) \leqq f(\alpha_1, \cdots, \alpha_k)$. *B. N. Moyls.*

See also: Fields, Rings: Barnett and Abian. **General Theory of Numbers:** Gyires. **Geometries, Euclidean and Other:** Vivier; Ruse. **Differential Geometry:** Santaló; Reichardt. **Programming, Resource Allocation, Games:** Braumann.

Polynomials

Knapowski, S. Certain theorems, concerning irreducibility of polynomials. Prace Mat. **1** (1955), 272–275. (Polish. Russian and English summaries)

In this paper some theorems are proved, which permit us to state the irreducibility of certain algebraic equations when the order of the Galois group is known. I. Let $f(x)=0$ be an equation of degree n, its coefficients belonging to a perfect field K, and none of its zeros belonging to K. Let $\mathfrak{G}$ be the Galois group of this equation and $\overline{\overline{\mathfrak{G}}}=m$. Let every zero of the equation $f(x)=0$ be expressed by a rational function of each pair of zeros of this equation. If $m^2 \nmid n!$, then the equation is irreducible. II. Every equation of degree p (p being prime) whose Galois group is of order p is irreducible. III. Let $f(x)=0$ be a cyclic equation of degree p^m (p being prime). The necessary and sufficient condition for the irreducibility of the equation is: $\overline{\overline{\mathfrak{G}}}=p^m$ ($\mathfrak{G}$ is the Galois group of the equation). *Author's summary.*

Knapowski, S. A criterion of irreducibility of the equations of degree $p+1$. Prace Mat. **2** (1956), 170–171. (Polish. Russian and English summaries)

Let $f(x)=0$ be an equation of degree $p+1$ (p a prime number) whose coefficients are rational numbers. Suppose that each zero of the equation can be expressed rationally by each pair of the other zeros. Denote by m the order of the Galois group of the equation in the field R of rational numbers. Theorem: The necessary and sufficient condition of the irreducibility of the equation $f(x)=0$ in R is $m \geqq p+1$. *Author's summary.*

Ehrenfeucht, A. A criterion of indecomposability of polynomials. Prace Mat. **2** (1956), 167-169. (Polish. Russian and English summaries)

If a polynomial $W(x_1, x_2, \cdots, x_n)$ is a sum of polynomials of one variable and the greatest common divisor of the degrees of those polynomials is equal to 1, then the polynomial W is indecomposable in the field of complex numbers. The combinatorial proof is based on the fundamental theorem of algebra. *Author's summary.*

See also: Fields, Rings: Barnett and Abian. **Functions of Complex Variables:** Dobrzycki.

Partial Order, Lattices

See: Foundations, Theory of Sets, Logic: Łoś. **Algebra:** Nolin. **Groups and Generalizations:** Hintzen; Dragunova.

Fields, Rings

Andrunakievič, V. A. Modular ideals, radicals and semi-simplicity of rings. Uspehi Mat. Nauk (N.S.) **12** (1957), no. 3(75), 133–139. (Russian)

A ring without radical (in the sense of Jacobson) and satisfying the minimal condition for principal right modular ideals is a finite direct sum of total matrix rings over fields. A ring without radical (in the sense of Brown-McCoy and of Segal) and satisfying the minimal condition for principal ideals is a discrete direct sum of simple rings with identity. A ring without radical (in the sense of Brown-McCoy and of Segal) and satisfying the minimal condition for principal modular ideals is a finite direct sum of simple rings with identity. *R. A. Good.*

Barnett, I. A.; and Abian, S. Some properties of skew-symmetric elements of a ring. Proc. Cambridge Philos. Soc. **53** (1957), 549–553.

Let $\mathscr{R}$ be a ring with 1. For all $A \in \mathscr{R}$, let the mapping $A \to A'$ be an involutory anti-automorphism; i.e., a 1-1 mapping of $\mathscr{R}$ onto itself such that $(A+B)'=A'+B'$, $(AB)'=B'A'$, $(A')'=A$. Under such a mapping the element A is called symmetric if $A'=A$ and skew-symmetric if $A+A'=0$. The following theorem is proved: Let $A \in \mathscr{R}$; if for some symmetric C, $A-C$ has an inverse B, then A is skew-symmetric if and only if $2B'CB+B'+B=0$. Some consequences of this result are given in the cases that (i) $\mathscr{R}$ is the ring of real n-square matrices and the mapping is transposition, (ii) $\mathscr{R}$ is an algebra $\mathfrak{a}$ over the real numbers of integral operators of a special type, (iii) $\mathscr{R}$ is a ring of matrices with elements in $\mathfrak{a}$. *B. N. Moyls.*

Bourne, Samuel; and Zassenhaus, Hans. On a Wedderburn-Artin structure theory of a potent semiring. Proc. Nat. Acad. Sci. U.S.A. **43** (1957), 613–615.

Throughout this review S will denote a semiring with an additive identity 0 such that $0 \cdot s = s \cdot 0 = 0$ for all $s \in S$. If S_1 and S_2 are subsemirings of S such that $S_1 \cap S_2 = (0)$, every element of S can be written uniquely as the sum of an element of S_1 and an element of S_2, and $s_1+s_2=s_2+s_1$ for all $s_1 \in S_1$, $s_2 \in S_2$, then S is said to be the strong direct sum of S_1 and S_2. [For other definitions, see same Proc. **42** (1956), 632–638; MR **18**, 188.] The main theorem of the paper is the following. If S is a potent semiring with identity in which each two-sided ideal contains a minimal left and a minimal right ideal, and S is a strong direct sum of minimal right ideals, then S is a strong direct sum of semirings which are isomorphic to semirings of matrices over division semirings. *M. Henriksen.*

Kleinfeld, Erwin. Assosymmetric rings. Proc. Amer. Math. Soc. **8** (1957), 983–986.

A nonassociative ring is called assosymmetric if it satisfies the condition that $(P(x), P(y), P(z))=(x, y, z)$, for every permutation P of x, y, z. The main result is that if R is an assosymmetric ring whose characteristic is not 2 or 3 without ideals $I \neq 0$ such that $I^2=0$, then R is associative. Examples of assosymmetric rings which are not associative are given and are used to show that the restriction on the characteristic in the main result is necessary. *L. A. Kokoris* (New Haven, Conn.).

Maury, Guy. Théorèmes de transfert de certaines propriétés de l'anneau $A[\theta]$, extension simple entière de A. C. R. Acad. Sci. Paris **245** (1957), 265-267.

Let B be a commutative ring, A a subring of B, θ an element of B integral over A and $f(x)$ of lowest degree in $A[x]$ with $f(\theta)=0$. By a P_i-ring we understand a local ring (when $i=1$), a regular local ring (when $i=2$), or an integral domain without 0-divisors which is integrally closed in B (when $i=3$). Necessary and sufficient conditions imposed on $f(x)$ are given for $A[\theta]$ to be a P_i-ring if A is a P_i-ring. *F. J. Terpstra* (Pretoria).

Bhattacharya, P. B. The Hilbert function of two ideals. Proc. Cambridge Philos. Soc. **53** (1957), 568-575.

Results of van der Waerden [Wetensch. Amsterdam, Proc. Akad. **31** (1928), 749–770] for bihomogeneous ideals of polynomials over a field are extended to the case of bihomogeneous ideals in polynomial rings over an Artin ring. These are applied (à la Samuel) to the study of pairs of $\mathfrak{m}$-primary ideals in a local ring Q with maximal ideal $\mathfrak{m}$. The main result is to the effect that, if $\mathfrak{q}_1$ and $\mathfrak{q}_2$ are two such ideals, then for large values of s and t the length of the ideal $\mathfrak{q}_1^s \cdot \mathfrak{q}_2^t$ is a polynomial in s and t of total degree equal to the dimension of Q. *H. T. Muhly.*

Nagata, Masayoshi. Note on a paper of Samuel concerning asymptotic properties of ideals. Mem. Coll. Sci. Univ. Kyoto. Ser. A. Math. **30** (1957), 165-175.

Soient R un anneau noethérien, $\mathfrak{a}$ et $\mathfrak{b}$ deux idéaux non nilpotents de R ayant le même radical. Pour tout entier n il existe un plus grand entier m tel que $\mathfrak{a}^n \subset \mathfrak{b}^m$ et un plus petit entier q tel que $\mathfrak{a}^n \supset \mathfrak{b}^q$; notons ces entiers $v_{\mathfrak{b}}(\mathfrak{a}, n)$ et $w_{\mathfrak{b}}(\mathfrak{a}, n)$; on sait que les suites $n^{-1}v_{\mathfrak{b}}(\mathfrak{a}, n)$ et $n^{-1}w_{\mathfrak{b}}(\mathfrak{a}, n)$ ont des limites $l_{\mathfrak{b}}(\mathfrak{a})$ et $L_{\mathfrak{b}}(\mathfrak{a})$ [Samuel, Ann. of Math. (2) **56** (1952), 11–21; MR **14**, 128]. En réponse à les problèmes posés par le rapporteur [loc. cit.] l'auteur montre que ces limites sont des nombres rationnels et que les déviations $v_{\mathfrak{b}}(\mathfrak{a}, n) - n \cdot l_{\mathfrak{b}}(\mathfrak{a})$ et $w_{\mathfrak{b}}(\mathfrak{a}, n) - n \cdot L_{\mathfrak{b}}(\mathfrak{a})$ sont majorées. Généralisant un résultat de Muhly [ibid. **60** (1954), 576–577; MR **16**, 213] au cas non-intègre il montre que deux idéaux $\mathfrak{a}$ et $\mathfrak{b}$ sont asymptotiquement équivalents si et seulement si chacun est entier sur l'autre. Il généralise aussi la "cancellation law" du rapporteur [loc. cit.]. Pour arriver à ces résultats, l'auteur généralise aux anneaux avec diviseurs de zéro la notion d'idéal complet (i.e., d'idéal défini par des conditions valuatives), et la méthode des transformations quadratiques; ainsi les limites $l_{\mathfrak{b}}(\mathfrak{a})$ et $L_{\mathfrak{b}}(\mathfrak{a})$ sont caractérisées en termes de valuations. Ce travail contient aussi quelques contre exemples (l'un montre que l'équivalence asymptotique n'est pas compatible avec l'intersection des idéaux), des remarques sur la dépendance intégrale des idéaux dans le cas non-noethérien, et d'autres sur le passage aux anneaux gradués associés. *P. Samuel* (Clermont-Ferrand).

See also: Algebras: Širšov.

Algebras

Yoshii, Tensho. Supplements and corrections to my paper: "On algebras of bounded representation type". Osaka Math. J. **9** (1957), 67-85.

The author supplements with some further details his proof of the main theorem of his cited paper [same J. **8** (1956), 51–105; MR **18**, 462]. Supplementary remarks are given also to his paper in Proc. Japan Acad. **32** (1956), 441–445 [MR **18**, 462]. *T. Nakayama* (Nagoya).

Rosenberg, Alex; and Zelinsky, Daniel. Tensor products of semiprimary algebras. Duke Math. J. **24** (1957), 555–559.

Let SP_1 mean the class of algebras over a field F which are "semiprimary" in the sense that the residue-algebra modulo some quasiregular ideal is semisimple (meaning semisimple with minimum condition). Replacing "quasiregular" with "nil" and "nilpotent" we obtain SP_2 and SP_3, respectively. The paper proves that if A, B are algebras over F and $A \otimes B \in SP_i$ $(i=1, 2, 3)$ then either $A \in SP_i$ or B is quasiregular, nil or nilpotent respectively. Further, if $A \otimes A^* \in SP_1$ (A^* being the opposite of A) and if C is a simple component of the (semisimple) residue-algebra of $A \otimes A^*$ modulo its radical, then C is finite over its center, Z, and Z is an algebraic extension of F whose maximal separable subfield is finite over F, and if $A \otimes A^* \in SP_3$ then A modulo its radical is finite over F; a corollary is that if $A \otimes B$ satisfies minimum condition for left ideals whenever B does, then A is finite over F. *T. Nakayama* (Nagoya).

Mitas, Günter. Zur Strukturtheorie separabler Algebren. J. Reine Angew. Math. **198** (1957), 1–6.

The following weaker form of a theorem of Wedderburn is proved: Every separable algebra A over an "explicitly known field Ω" [defined in Mitas, same J. **197** (1957), 68–75; MR **19**, 11] is a direct sum of uniquely determined irreducible separable algebras A_i whose centers are separable over a scalar extension of Ω. These algebras can be constructed exactly in a finite number of steps if every polynomial over Ω which is relatively prime to its derivative can be written in a finite number of steps as a product of irreducible polynomials. These results are proved by methods which avoid the use of the radical. *L. A. Kokoris* (New Haven, Conn.).

Lyu, Šao-Syuè. On the splitting of locally finite algebras. Mat. Sb. N.S. **39(81)** (1956), 385-396. (Russian)

The principal Wedderburn theorem for an associative finite algebra over a field (including both the existence and the uniqueness aspects of the decomposition) has been much generalized. Some authors have treated alternative or Lie or Jordan algebras, others have considered certain algebras of infinite rank. The goal of this paper is a unified version of the Wedderburn structure theory which subsumes as special cases all the above situations. The generality achieved is also applicable to various types of locally finite algebras. *R. A. Good.*

Širšov, A. I. On some non-associative nil-rings and algebraic algebras. Mat. Sb. N.S. **41(83)** (1957), 381–394. (Russian)

For a historical summary of the Kuroš ring-theory problem [Izv. Akad. Nauk SSSR Ser. Mat. **5** (1941), 233–240; MR **3**, 194], see the review of the Levitzki contribution [Bull. Amer. Math. Soc. **52** (1946), 1033–1035; MR **8**, 435]. The author studies the analogous problem in certain non-associative situations. An enveloping associative ring of a semi-special nil-J-ring of bounded index with finitely many generators and not containing elements of additive order two is nilpotent. Every enveloping associative algebra of a special algebraic J-algebra of bounded degree over a field of characteristic not two is locally finite. An alternative nil-ring of bounded index and not containing elements of additive order two is locally nilpotent. Every alternative algebraic algebra of bounded degree over a field of characteristic not two is locally finite. *R. A. Good* (College, Park, Md.).

Taft, Earl Jay. The Whitehead first lemma for alternative algebras. Proc. Amer. Math. Soc. **8** (1957), 950–956.

Results of the reviewer concerning alternative algebras of characteristic 0 [Trans. Amer. Math. Soc. **72** (1952), 1–17; MR **13**, 527] are extended to characteristic $\neq 2, 3$. Proofs involve the alternative theory directly, rather than the earlier reliance upon Jacobson's results for Jordan algebras of characteristic 0.

It is first proved that every derivation of a Cayley algebra C over a field Φ of characteristic $\neq 2, 3$ has the form $\sum_i D_{x_i,z_i}$ for x_i, z_i in C, where $D_{x,z}=[R_x, R_z]+[R_x, L_z]+[L_x, L_z]$. Theorem: Let A be a (finite-dimensional) separable alternative algebra over Φ, and let (R, L) be an alternative representation of A acting in the bimodule M. Let f be a one-cocycle of A into M; that is, a linear transformation of A into M satisfying $f(ab)=f(a)b+af(b)=f(a)R_b+f(b)L_a$ for all a, b in A. Then there exist an element g in the nucleus of M, and elements x_i in A, z_i in M, such that $f(a)=[a, g]+a\sum_i D_{x_i,z_i}$. (If Φ is of characteristic 0, then g may be taken as zero.) Equivalently, if A is a separable subalgebra of an alternative algebra B over Φ, then any derivation of A into B can be extended to an inner derivation of B which has the form $R_g-L_g+\sum_i D_{x_i,y_i}$ for g in the nucleus of B, x_i, y_i in B. (If Φ is of characteristic 0, then g may be taken as zero.)

R. D. Schafer (Storrs, Conn.).

Groups and Generalizations

★ **Lugowski, Herbert; und Weinert, Hanns Joachim. Grundzüge der Algebra, Teil I. Allgemeine Gruppentheorie.** Mathematisch-Naturwissenschaftliche Bibliothek, 9. B. G. Teubner Verlagsgesellschaft, Leipzig, 1957. v+234 pp. DM 10.00.

This volume is the first of an intended three-volume presentation of modern algebra; volumes 2 and 3 will have the titles "Allgemeine Ring- und Körpertheorie" and "Auflösungstheorie algebraischer Gleichungen". The style is detailed and suitable for self-instruction. There are many problems with complete solutions. The four chapter headings are: Grundbegriffe der Gruppentheorie; Permutations- und Transformationsgruppen; Homomorphe Abbildungen; Einige wichtige Strukturaussagen über Gruppen.

Crouch, R. B.; and Scott, W. R. Normal subgroups of monomial groups. Proc. Amer. Math. Soc. **8** (1957), 931–936.

Determination of all normal subgroups of the group of those monomial substitutions of a fixed infinite set, with coefficients in a fixed group, which (i) have only a finite number of coefficients different from the identity, and (ii) have underlying permutations which move a set of power less than a fixed cardinal. *Graham Higman.*

Fuchs, L. Über universale homomorphe Bilder und universale Untergruppen von abelschen Gruppen. Publ. Math. Debrecen **5** (1957), 185-196.

For G an abelian group, a group U is called a universal homomorphic image of G if 1) U is a homomorphic image of G, and 2) every homomorphic image of G is isomorphic to a subgroup of U. Dually, a group Z is called a universal subgroup of G if 1) Z is a subgroup of G, and 2) every subgroup of G is a homomorphic image of Z. It is proved that a universal homomorphic image is possessed by any torsion abelian group and by any mixed abelian group whose torsion-free factor group has infinite rank. The actual theorem is: if G is an abelian group with torsion subgroup T and p-primary part T_p, then G has a universal homomorphic image if and only if either rank (G/T) is zero or infinite, or rank $(p^n \cdot T_p)$ is infinite for all n and p and G/T is a direct sum of rank one groups whose types are linearly ordered. It is also proved that G has a universal subgroup if and only if either rank (G/T) is infinite or G/T is free and, for every prime p, $\min_n(\text{rank}(p^n \cdot T_p))$ is zero or infinite. *D. K. Harrison* (Providence, R.I.).

Kargapolov, M. I. Factorization of Π-separable groups. Dokl. Akad. Nauk SSSR (N.S.) **114** (1957), 1155–1157. (Russian)

A subgroup $\mathfrak{H}$ of a group $\mathfrak{G}$ is said to be closed relative to a set Π of primes provided each Sylow p-subgroup of $\mathfrak{H}$, where $p \in \Pi$, is also a Sylow subgroup of $\mathfrak{G}$. Let $\Pi(\mathfrak{H})$ denote the set of primes which are divisors of the orders of elements of the subgroup $\mathfrak{H}$. If Π is a set of primes, a collection of pairwise permutable subgroups $\mathfrak{P}_1, \mathfrak{P}_2, \cdots, \mathfrak{P}_n, \cdots$ of a locally finite group $\mathfrak{G}$ is called a Π-separable basis for $\mathfrak{G}$ provided these three conditions hold: (1) $\mathfrak{G}=\{\mathfrak{P}_1, \cdots, \mathfrak{P}_n, \cdots\}$; (2) for each n, the subgroup $\mathfrak{P}_n$ is closed relative to $\Pi \cap \Pi(\mathfrak{P}_n)$; (3) if $\Pi \cap \Pi(\mathfrak{G})=\{p_1, \cdots, p_n, \cdots\}$, then $\Pi \cap \Pi(\mathfrak{P}_n)=\{p_n\}$. Every locally Π-separable group possessing a normal series with finite factors and satisfying the minimal condition for subgroups has at least one Π-separable basis.

R. A. Good (College Park, Md.).

Hall, Marshall, Jr. Solution of the Burnside problem rof exponent 6. Proc. Nat. Acad. Sci. U.S.A. **43** (1957), 751–753.

An outline of the proof, to appear in detail elsewhere, that a finite y generated group of exponent 6 is finite. Most of the intermediate lemmas are statements that must be true if the main theorem is to have a chance, in view of the known structure of finite groups of exponent 6 [P. Hall and Higman, Proc. London Math. Soc. (3) **6** (1956), 1–42, pp. 1–20; MR **17**, 344]. An exception is the result that a group of exponent 6, generated by 4 elements, any 3 of which generate a group of exponent 3, is itself of exponent 3. *Graham Higman* (Oxford).

★ **Kippels, Herbert. Das Problem der Isomorphie bei situationsgleichen Gruppen von endlicher Ordnung.** Inaugural-Dissertation. Zentral-Verlag für Dissertationen Triltsch, Düsseldorf, 1955. 40 pp.

Two groups G and G^* are said to be "similarly situated" if there exists a "U-mapping" α of the subgroups U_i of G onto the subgroups U_i^* of G^*, which (a) preserves the orders of subgroups, (b) preserves inclusion relations among subgroups, and (c) takes conjugate subgroups of G into conjugate subgroups of G^*. Rottlaender [Math. Z. **28** (1928), 641–653] and Baer [S.-B. Heidelberg. Akad. Wiss. Math.-Nat. Kl. **1933**, no. 2, 12–17] had shown that similar situation implies isomorphism for abelian and hamiltonian groups, but not for certain metacyclic groups. The author extends a U-mapping α to a single valued E-mapping ε of elements in all possible ways, and illustrates a non-isomorphic mapping with a non-cyclic group of order 25. Any situation-preserving U-mapping preserves normalizers, centers, the order of nilpotency of nilpotent groups, the commutator series of finite groups, and the rank of metacyclic groups. The author then studies nilpotent p-groups G of second class, with center

Z such that G/Z is abelian, uses commutator relations such as $(AB, C)=(A, C)(B, C)$, a obtains and unique representation of its elements in terms of certain generators U_i of U and UV_j of G/U, where $K\subseteq U\subseteq Z$. The group G is determined from the U_i and V_j if the expressions in terms of U_i for the commutators (V_j, V_k) and for the least powers of V_j belonging to U are given. The principal structure theorem is as follows. Two similarly situated nilpotent groups G and G^* of second class are isomorphic if and only if there exists an E-mapping of elements establishing an isomorphism between subgroups U and U^*, where $K\subseteq U\subseteq Z$, which preserves powers of elements and commutators of elements. Among nilpotent groups of class two are the "almost abelian" groups for which K is of order 2. These are uniquely determined by their "situation plan". *J. S. Frame* (East Lansing, Mich.).

Knapowski, S. A theorem of the theory of finite groups. Prace Mat. **2** (1956), 165–166. (Polish. Russian and English summaries)

Denote by S_n the group of all the substitutions of n elements. Let positive integers $n\geqq 2$, m satisfy one of the following two conditions: (1) $n\nmid m$, each prime divisor of m is greater than $\frac{1}{2}n$; (2) $m=p^a$ (p a prime number), $p\nmid n$. If there exists a group $G_n\subset S_n$, $\bar{\bar{G}}=m$, then there exists a group $\Gamma_n\subset S_{n-1}$, isomorphic with G. *Author's summary.*

Želudev, I. S. Symmetry of scalars, vectors and tensors of second rank. Akad. Nauk SSSR. Kristallografiya **2** (1957), 207-216. (Russian)

This paper gives, in the main, an account of ideas put forward by A. V. Šubnikov [Izv. Akad. Nauk SSSR. Ser. Fiz. **13** (1949), 347–375, 376–391; MR **11**, 318]. At the end of the paper the symmetries of bivalent tensors are tabulated into seventeen groups as follows.

		Symmetries
1.	Scalar	$\infty/\infty\cdot m$
2.	Pseudoscalar	∞/∞
3.	Polar vector	$\infty\cdot m$
4.	Axial vector	$\infty:m$
5, 6.	Polar tensor	$m\cdot\infty:m$, $m\cdot 2:m$
7, 8, 9.	Axial tensor	$\infty:2$, $2:2$, $\bar{4}\cdot m$
10, 11.	Combinations of polar tensor and axial vector	$2:m$, $\bar{2}$
12–16.	Combinations of axial tensor and polar vector	∞, 2, 1, $2\cdot m$, m
17.	Combinations of axial tensor and axial vector	$\bar{4}$

("Scalar" means a bivalent tensor with $a_{jk}=\delta_{jk}$, etc.)

Added is a neat diagram with a geometrical representation of each type. *D. J. Struik.*

Huppert, Bertram. Lineare auflösbare Gruppen. Math. Z. **67** (1957), 479-518.

This paper studies a soluble linear group G, with particular attention to the limits imposed on its structure by the dimension n of the space on which it acts. For instance it is proved that the derived length of G cannot exceed $1+7\log_2 n$, or, if G is completely reducible, $6\log_2 n$; bounds which are at least of the right order of magnitude. Such results have applications to the study of the abstract soluble group G. For if r is the upper bound of the exponents r_i in the orders $p_i^{r_i}$ of the chief factors of G, the automorphism groups induced in its chief factors by G are linear groups of degree at most r. Since the intersection of the centralisers of the chief factors is nilpotent, the kth derived group of G is nilpotent for some k not greater than $6\log_2 r$.

The main results are naturally proved by induction on the degree n. If G is reducible it is clear how to proceed. If not, consider a normal subgroup N of G. N is completely reducible. If its irreducible components are not all equivalent, G is imprimitive, and is therefore an extension of a permutation group of degree at most n by what is effectively a linear group of degree less than n. Since the permutation group, considered as a linear group over the complex numbers, has irreducible components of degree at most $n-1$, we are again in a position to use induction. If the irreducible components of N, though all equivalent, have a degree strictly between 1 and n, then every element of G can be written as a Kronecker product, and hence G can be expressed as a subgroup of a central product of groups of smaller degrees, again permitting induction to be applied. There remains the case when all reducible normal subgroups belong to the centre of G, so that in particular, if the ground field is algebraically closed, all abelian normal subgroups belong to the centre. This case requires an investigation of a minimal non-abelian normal subgroup N of G. Such a group is a q-group for some prime q, and its factor group by its centre Z is elementary abelian of order q^{2m}, where $q^m=n$. Moreover the centraliser of N/Z is metabelian. Since, modulo this centraliser, G is a linear group of degree $2m$, we are again in a position to apply induction, unless $2m\geqq n=q^m$, that is, unless $n=2$ or 4, cases which have to be dealt with separately.

The author proves a great many results of considerable variety and interest; but I think the scope of the paper is better indicated by the above outline, than by any reasonably short selection of results. *Graham Higman.*

Shepperd, J. A. H. Betweeness groups. J. London Math. Soc. **32** (1957), 277-285.

Auf Grund früherer Untersuchungen [dasselbe J. **31** (1956), 240–248; MR **18**, 61] wählt Verfasser einen Zwischenbegriff, so daß eine B-Menge, d.h. eine Menge mit Zwischenrelation, entweder linear ist (d.h. die Zwischenrelation ist von einer linearen Ordnung induziert) oder genau aus vier Elementen mit zyklischer Zwischenrelation besteht. Außer 4-elementigen Gruppen sind alle B-Gruppen, d.h. Gruppen die B-Mengen sind und deren Zwischenrelation invariant ist bei Links- oder Rechtsmultiplikation, unendlich und enthalten höchstens ein Element endlicher Ordnung, nämlich der Ordnung 2. Verfasser kann daher die Strukturprobleme der B-Gruppen vollständig auf die der geordneten Gruppen zurückführen. *P. Lorenzen* (Kiel).

★ **Hintzen, Josef. Ein System von unabhängigen Axiomen für Halbgruppen mit eindeutigen Halbprimfaktorzerlegungen.** Inaugural-Dissertation zur Erlangung des Doktorgrades der Mathematisch-Naturwissenschaftlichen Fakultät der Universität zu Köln, 1957. i+21 pp.

A commutative semigroup in which $a|b$ (a divides b) and $b|a$ together imply that $a=b$ is called a holoid. A holoid is a partially ordered set under the ordering: $a\leq b$ if $a|b$. If a holoid has an identity element then it (and it alone) is said to be a trivial divisor of any element. An element p of a holoid is said to be semiprime if in any factorization of p one of the factors must be p. The identity element (if there is one) is said to be a trivial semiprime element. The author investigates the properties of factorization of elements in holoids which satisfy

the four conditions: (i) if $a_1 \geqq a_2 \geqq a_3 \geqq \cdots \geqq a_r \geqq \cdots$ is a sequence of elements none of which is semiprime, then for some n, $a_m = a_n$ for all $m \geqq n$; (ii) any divisor of a semiprime element is semiprime; (iii) if two semiprime elements have a common non-trivial divisor then one of them divides the other; (iv) if $d|ab$ and d and a are relatively prime (i.e., have at most a trivial common divisor) then $d|b$. An element p is said to be a maximal semiprime divisor of x if $p|x$ and there is no semiprime element strictly greater than p which also divides x. The main theorem of the paper (also proved under weaker hypotheses) states that any element x in a holoid satisfying conditions (i)–(iv) has a unique finite set of distinct maximal semiprime divisors and that if $x_1, x_2, \cdots, x_n$ are the maximal semiprime divisors of x then $x = x_1^{r_1} x_2^{r_2} \cdots x_n^{r_n}$, where $r_i \geqq 1$ and where the factors $x_i^{r_i}$ are uniquely determined (except for order). It is shown that the unique factorization theorems of F. Klein-Barmen for A-classes and for B-classes [Math. Z. **37** (1933), 39–60], of A. H. Clifford for holoids satisfying conditions A, B and C (loc. cit.) [Ann. of Math. (2) **39** (1938), 594–610], of R. P. Dilworth for distributive lattices [ibid. **41** (1940), 771–777; MR **2**, 120] are all special cases of the author's theorems. {Erratum: the word 'maximalen' in the statement of Theorem 7 (p. 12, line 12) should be omitted.}

G. B. Preston (New Orleans, La.).

Dragunova, T. E. Structure of right ideals of a semigroup with unit element and with left cancellation. Uspehi Mat. Nauk (N.S.) **12** (1957), no. 4(76), 285-288. (Russian)

In a complete distributive lattice L, let A be the collection of those elements a such that L is isomorphic to the section determined by the element a. Call L semi-uniform provided, first, each of its elements, other than 0, is expressible as a sum of members from A and, second, whenever $k \in L$ is so expressible in two different ways, $k = \sum a_i = \sum b_j$ with $a_i \in A$ and $b_j \in A$, then to each i corresponds some j such that $a_i \leqq b_j$. Given any complete distributive lattice L that is semi-uniform, there exists a semigroup S with identity and satisfying the left cancellation law such that L is isomorphic to the lattice of all right ideals of S. Conversely, given any semigroup S with identity and satisfying the left cancellation law, the lattice of all right ideals in S is a semi-uniform complete distributive lattice. Note that the empty subset of a semigroup is considered as a right ideal. Also generalized is Theorem 2.2 in the paper by Rees [Quart. J. Math. Oxford Ser. **19** (1948), 101–108; MR **9**, 567]. *R. A. Good.*

Drbohlav, Karel. Gruppenartige Multigruppen. Czechoslovak Math. J. **7(82)** (1957), 183-190. (Russian summary)

The author examines multigroupoids, that is, systems with a multivalued operation. The main result is a characterisation of those multigroups (gruppenartige Multigruppen) which are isomorphic to the multiplicative system of cosets for a group G with respect to a subgroup H. Let M be a multigroup with a scalar right unit j so that $jx \supset x$, $xj = x$ for $x \in M$. A permutation π of M is essential when $\pi(ab) = \pi(a) \cdot b$. The essential permutations form a group Γ and those with $\pi(j) = j$ a subgroup Δ. Then M is isomorphic to a coset expansion, namely Γ/Δ, if and only if Γ is transitive on M and Δ transitive for each product set jx. *O. Ore* (New Haven, Conn.).

See also: Structure of Matter: Judd. **Quantum Mechanics:** Duculot.

THEORY OF NUMBERS

General Theory of Numbers

Zeckendorf, E. Les équations quadratiques. Bull. Soc. Roy. Sci. Liège **26** (1957), 112–122.

The author considers the recurring series (1) $t_{x+2} = nt_{x+1} + mt_x$, $t_0 = a$, $t_1 = b$ and $\tau_{x+2} = n\tau_{x+1} + m\tau_x$, $\tau_0 = 2b - na$, $\tau_1 = 2ma + nb$ ($\tau_x = mt_{x-1} + t_{x+1}$). He proves that $\tau_x^2 - (n^2 + 4m)t_x^2 = 4k_1(-m)^x$, $k_1 = b^2 - ma^2 - nab$, which is an immediate generalization of the well-known formula in case $a = 0$, $b = 1$. He then treats "quadratic equations", i.e., equations of the types (2) $A^2 - (n^2 + 4m)B^2 = 4k_1(-m)^x$ for n odd and the derived equations $A^2 - (v^2 + m)B^2 = k_1(-m)^x$, $n = 2v$, $A^2 + AB - (v^2 + v + m)B^2 = k_1(-m)^x$, $n = 2v + 1$ for given integer values of n, m, k_1 and x. As an example we mention that for a given numerical value of a satisfying $k_1 = b^2 - ma^2 - nab$, $m \neq \pm 1$, two solutions in positive integers A, B of (2) are given by (1), viz., the numerical values of (τ_x, t_x) and those of $(m^x \tau_{-x}, m^x t_{-x})$. Nothing is said about the general solution of this equation.

W. Ljunggren (Oslo).

Godwin, H. J. A note on $x^3 + y^3 + z^3 = 1$. J. London Math. Soc. **32** (1957), 501–503.

This paper supplements a note by the reviewer on the diophantine equation in the title. The sets of solutions for which $\alpha = (1 - z)/(x + y)$ has a fixed value are recursively related via the theory of the Pell equation to the indeterminate equation

$$3v^2 - (4\alpha^3 - 1)u^2 = 4(1 - \alpha^3).$$

The author considers the general case of rational α which lead to integer solutions (x, y, z). For the case $\alpha = 4$ a new set of solutions represented by $x = 67402$, $y = -83802$, $z = 65601$ is found. For $\alpha = 7$, four new sets are added to the four already known. *D. H. Lehmer.*

Ginatempo, Nicola. Su un teorema di Betti. Boll. Un. Mat. Ital. (3) **12** (1957), 312–315.

The general solution of the linear diophantine equation in n variables had already been stated by Betti in terms of parameters involving a skew-symmetric matrix. The author finds the same result by the method of induction [see Barnett and Mendel, Amer. Math. Monthly **49** (1942), 157–170, p. 158; MR 3, 268]. *I. A. Barnett.*

Piehler, Joachim. Bemerkungen zur Verteilung der kubischen Reste. Math. Ann. **134** (1957), 50–52.

For a prime $p \equiv 1 \pmod 3$, the cubic residues mod p constitute a subgroup, R_0, of index 3, of the group of residue classes modulo p which consist of integers $\not\equiv 0 \pmod p$. Let R_1, R_2 denote the two cosets of this subgroup which comprise the cubic non-residues. The author defines h_{ijk} (where $i, j, k = 0, 1, 2$) to be the number of residue classes mod p in R_k which, for a given fixed $z_j \in R_j$, are representable in the form $z_j + w$, $w \in R_i$. These h_{ijk} are independent of the particular choice of z_j in R_j. Setting $\lambda = (p - 1)/3$, the h_{ijk} are determined as the unique solution to the inequality $0 \leqq h_{ijk} \leqq \lambda$ and the following congruences. Case I, $i + j + k \equiv 0 \pmod 3$: (a) for $i = j = k$,

$$h_{ijk}\equiv\frac{1}{9}\left[3\lambda-7-\left(\frac{2\lambda}{\lambda}\right)\right] \pmod{p};$$

(b) for i, j, k all different,

$$h_{ijk}\equiv\frac{1}{9}\left[3\lambda+2-\left(\frac{2\lambda}{\lambda}\right)\right] \pmod{p}.$$

Case II, $i+j+k\equiv\nu\not\equiv 0 \pmod 3$ (i.e. $\nu=1$ or 2), $z_\nu\in R_\nu$:

$$h_{ijk}\equiv\frac{1}{9}\left[3\lambda-1-\left(\frac{2\lambda}{\lambda}\right)z_\nu{}^{\lambda}\right] \pmod{p}.$$

H. N. Shapiro (New York, N.Y.).

Carlitz, L. A theorem of Dickson on nonvanishing cubic forms in a finite field. Proc. Amer. Math. Soc. **8** (1957), 975–977.

The theorem in the title is as follows. Let $f(x, y, z)$ be a homogeneous cubic polynomial with coefficients in GF(q), $q=p^n$, that vanishes in GF(q) only for $x=y=z=0$. Then if $p>2$, $q\geqq 13$, $f(x, y, z)$ is the norm of a linear form $\alpha x+\beta y+\gamma z$, where α, β, γ are in GF(q^3) and are linearly independent with respect to GF(q). Dickson's proof [Bull. Amer. Math. Soc. **14** (1908), 160–169] is inadequate. In this note the author sketches another proof. The converse of the theorem is evidently true. Possibly the theorem holds for $p=2$, but the author's proof does not apply in that case. *A. L. Whiteman* (Los Angeles, Calif.).

Gyires, Béla. Über eine Verallgemeinerung des Smithschen Determinantensatzes. Publ. Math. Debrecen **5** (1957), 162–171.

Let r be a non-negative integer, and define

$$\phi_r(k)=\begin{cases}1 & (k=1),\\ k^{r+1}\prod_{p\mid k}\left(1-\frac{1}{p^{r+1}}\right) & (k\geqq 2),\end{cases}$$

so that $\phi_0(k)=\varphi(k)$. The author proves that if D_{r+1} is the $n\times n$ determinant whose (i, j)th element is $(i, j)^{r+1}$, then

$$D_{r+1}=\phi_r(1)\phi_r(2)\cdots\phi_r(n).$$

The case $r=0$ is a well-known result of H. J. S. Smith. The author actually evaluates a more general determinant containing D_{r+1} as a special case. *M. Newman.*

★ **Yahya, Q. A. M. M. Complete proof of Fermat's last theorem. With a foreword by Dr. Razi-Ud-Din Siddiqui.** Available from the author, Pakistan Air Force, Kohat, West Pakistan, 1958. 14 pp. Mimeographed appendix, 3 pp.

See also: Theory of Algebraic Numbers: O'Meara.

Analytic Theory of Numbers

Hua, Lo-Kèn. Introduction to the theory of vector modular forms. Akad. Nauk Azerbaĭdžan. SSR. Trudy Inst. Fiz. Mat. **3** (1948), 32–43. (Russian. Azerbaijani summary)

Mikolás, Miklós. Mellinsche Transformation und Orthogonalität bei $\zeta(s, u)$. Verallgemeinerung der Riemannschen Funktionalgleichung von $\zeta(s)$. Acta Sci. Math. Szeged **17** (1956), 143–164.

Hurwitz's function $\zeta(s, u)=\sum_{n=0}^{\infty}(u+n)^{-s}$ ($\Re(s)>1$, $u>0$) can be continued into the whole s-plane as a meromorphic function with one simple pole at $s=1$. Put $\xi(s, u)=\zeta(s, v)$, where $v\equiv u \pmod 1$, $0<v\leqq 1$. The Mellin transform

$$\mathfrak{M}(s, z)=\int_0^\infty u^{z-1}\xi(s, u)du, \quad \max(0, \Re(s))<\Re(z)<1,$$

is proved to satisfy the relation

$$\mathfrak{M}(1-s, z)=\mathfrak{M}(1-z, s)=$$
$$=2(2\pi)^{-s-z}\Gamma(s)\Gamma(z)\cos\frac{\pi(s-z)}{2}\zeta(s+z),$$

which for $\Re(s)\geqq 1$, $z\to+0$, becomes the functional equation of the Riemann ζ-function. In the opposite direction,

$$\zeta(1-s, u)-u^{s-1}=$$
$$\frac{1}{2\pi i}\int_{\alpha-\infty i}^{\alpha+\infty i}u^{-z}\cdot 2(2\pi)^{-s-z}\Gamma(s)\Gamma(z)\cos\frac{\pi(s-z)}{2}\zeta(s+z)dz$$

if $0<u<1$, $0<\Re(s)<1$, $0<\alpha<\max(\Re(s), 1-\Re(s))$. The author proves further integral formulae; e.g.,

$$\int_0^1\zeta(s, u)\zeta(z, u)du=$$
$$(2\pi)^{s+z-2}\Gamma(1-s)\Gamma(1-z)\cos\frac{\pi(s-z)}{2}\zeta(2-s-z)$$

if $\max(0, \Re(s))+\max(0, \Re(z))<1$,

$$\int_0^1|\zeta(s, u)|^2du=2(2\pi)^{2(\sigma-1)}\cosh\pi\tau\cdot|\Gamma(1-s)|^2\zeta(2-2\sigma)$$
$$(s=\sigma+\tau i, \sigma<\tfrac{1}{2});$$

and he announces more general identities which appear in the paper reviewed below. *K. Mahler* (Manchester).

Mikolás, Miklós. Integral formulae of arithmetical characteristics relating to the zeta-function of Hurwitz. Publ. Math. Debrecen **5** (1957), 44–53.

The Hurwitz zeta function satisfies the formula

$$\zeta(1-s, x)=\frac{2\Gamma(s)}{(2\pi)^s}\left(\cos\frac{\pi s}{2}\sum_{m=1}^{\infty}\frac{\cos 2\pi mx}{m^s}+\sin\frac{\pi s}{2}\sum_{m=1}^{\infty}\frac{\sin 2\pi mx}{m^s}\right)$$

if $s=\sigma+i\tau$, $\sigma>0$, $0<x<1$. The author shows that $\zeta(1-s, x)\in L^2(0, 1)$ if $\sigma>\frac{1}{2}$. Considering the above as a Fourier expansion, he uses the Parseval formula to derive the identity

$$\int_0^1\zeta(1-s, \{au\})\zeta(1-s, \{bu\})du=2\Gamma(s)^2\frac{\zeta(2s)}{(2\pi)^{2s}}\frac{(a, b)^2}{[a, b]^2},$$

where a, b are positive integers, (a, b) their g.c.d., $[a, b]$ their l.c.m., and $\{x\}=x-[x]$ is the fractional part of x. A similar formula is derived with the factor $\zeta(1-s, \{bu\})$ replaced by its complex conjugate. The hypothesis $\sigma>\frac{1}{2}$ is essential because the integrals do not exist when $\sigma\leqq\frac{1}{2}$. *T. M. Apostól* (Pasadena, Calif.).

Carlitz, L. A formula connected with lattice points in a circle. Abh. Math. Sem. Univ. Hamburg **21** (1957), 87–89.

An asymptotic formula for $\sum_{n=1}^{N}r(n)(\log n-\log N)$ with error $O(N^{-\frac{1}{2}})$ has been proved by Cl. Müller [same Abh. **19** (1954), no. 1–2, 62–65; MR **15**, 940], by a method connected with partial differential equations. The present author gives a derivation on more classical lines, using partial summation and a summation formula for $r(n)$. The constant term appears as a series involving $r(n)$, in place of Müller's closed form $4\log\Gamma(\frac{1}{4})-\log(4\pi)$. There are a few misprints. *F. V. Atkinson* (Canberra).

See also: Special Functions: Christian.

Theory of Algebraic Numbers

Urazbaev, B. M. On the growth of the number of completely critical cyclic fields of degree l^3. Dokl. Akad. Nauk SSSR (N.S.) **113** (1957), 1222–1223. (Russian)

After citing the form for the discriminant of various types of abelian extension fields, the author is able to present an asymptotic formula for the number described in the title of the paper. [Cf. Urazbaev, Izv. Akad. Nauk Kazah. SSR **1952**, no. 116, Ser. Astr. Fiz. Mat. Meh. 1(6), 115–122; Dokl. Akad. Nauk SSSR (N.S.) **95** (1954), 935–938; MR **16**, 339; **15**, 937.] *R. A. Good.*

Cohn, Harvey; and Gorn, Saul. A computation of cyclic cubic units. J. Res. Nat. Bur. Standards **59** (1957), 155–168.

As a result of an EDVAC computation, this paper lists the components and traces of a unit and of its reciprocal, and the "structure constants" for each of the 45 cyclic cubic fields of discriminant l^2, where $l = 3m + 1$ is a prime < 500. The unit is the fundamental unit of the field, except that for l=373, 379, 463, the square of the fundamental unit is listed instead. The computation is described in considerable detail. The program ran to completion only for $l \leqq 37$ and l=67, terminating due to excessive round-off error in 11 cases and due to overflow (integers $\geqq 2^{44}$) in the remaining cases. In all but 18 cases, at least one unit was produced during the computation, and in these 18 cases units were hand-computed from the machine output. The units listed were shown to be fundamental by a comparison with a table of the traces of the fundamental units communicated by P. Swinnerton-Dyer. *J. L. Selfridge* (Los Angeles, Calif.).

O'Meara, O. T. Local characterization of integral quadratic forms by Gauss sums. Amer. J. Math. **79** (1957), 687–709.

This paper is concerned with the existence of a unimodular solution X to the n by n matrix equation $X^THX=G$, with H and G symmetric. The author deals with characterization of equivalence classes by means of Gaussian sums over local fields with a finite residue class field. He shows that Gaussian sums alone characterize the class of H when ord 2=0, while one must include the type as an invariant in the unramified case. In the ramified theory further conditions must be imposed. If, however, H is unitary, the Gauss sums form a complete set of invariants. *B. W. Jones* (Boulder, Colo.).

See also: Special Functions: Christian.

Geometry of Numbers

See: Functions of Complex Variables: Uchiyama.

ANALYSIS

★ **Фихтенгол'ц, Г. М. [Fihtengol'c, G. M.] Основы математического анализа. Том I. [Foundations of mathematical analysis. Vol. I.]** Gosudarstv. Izdat. Tehn.-Teor. Lit., Moscow, 1957. 440 pp. 9.75 rubles.

A stereotype reprinting of the book reviewed in MR **18**, 876.

Functions of Real Variables

★ **Buck, R. Creighton. Advanced calculus.** McGraw-Hill Book Company, Inc., New York-Toronto-London, 1956. viii+423 pp. $8.50.

This book represents a very unusual combination of precision, comprehensiveness and readability. There are 7 chapters: elementary topology, functions, integration, convergence, differentiation, applications to geometry and analysis, elements of differential geometry. There is an appendix on foundations of the number system. There are numerous well-chosen problems with answers and, in some cases, suggestions on the method of solution.

★ **Cogan, Edward J.; and Norman, Robert Z. Handbook of calculus, difference and differential equations.** Prentice-Hall Mathematics Series. Prentice-Hall, Inc., Englewood Cliffs, N. J., 1958. xii+263 pp. $4.50.

This volume is a rewritten version of a text originally composed at the suggestion of the Committee on the Undergraduate Program of the Mathematical Association of America. It is now intended also to supplement a basic calculus and differential equations course for engineering and science students. Its key feature is its tables, of solutions of differential equations, of differences, of antidifferences, of solutions of difference equations, etc.

Karapandjitch, Georges. Contribution au problème des quadratures. Bull. Soc. Math. Phys. Serbie **8** (1956), 117–120. (Serbo-Croatian summary)

If a particular solution y_0 of $y'+Py+Q=0$ is known and if the quadrature of P can be performed, then the quadrature of $Q \exp \int P dx$ can also be performed, since

$$\int Qe^{\int P dx}\,dx = k_0 - y_0 e^{\int P dx}$$

where k_0 is a constant of integration. Particular choices of P and Q lead to interesting quadratures.

Timan, A. F.; and Timan, M. F. On the relation between the moduli of smoothness of functions defined on the whole real axis. Dokl. Akad. Nauk SSSR (N.S.) **113** (1957), 995-997. (Russian)

Let $f \in L_p(-\infty, \infty)$ $(1 \leqq p \leqq \infty)$. The modulus of smoothness of f of order k is

$$\omega_k(f; t) = \sup_{|h| \leqq t} \left\{ \int_{-\infty}^{\infty} \left| \sum_{\nu=0}^{k} (-1)^{k-\nu} \binom{k}{\nu} f(x+\nu h) \right|^p dx \right\}^{1/p}.$$

The authors give the following estimate for ω_k in terms of ω_ν with $\nu > k$: for $0 < t \leqq \frac{1}{2}$,

$$\omega_k(f; t) \leqq C_{\nu,k} t^k \int_t^1 \int_{t_1}^2 \cdots \int_{t_{\nu-k-1}}^{\nu-k} t_{\nu-k}^{-\nu} \omega_\nu(f; t_{\nu-k})\, dt_1 \cdots dt_{\nu-k},$$

where $C_{\nu,k}$ is independent of f; p is the same on both sides. In particular,

$$\omega_k(f; t) \leqq C_k t^k \int_t^1 u^{-k-1} \omega_{k+1}(f; u)\, du.$$

Some corollaries are given. The authors state that the proofs depend on the following inequalities connecting ω_k and $A_\sigma(f)$, the best L_p approximation to f by entire functions of exponential type σ:

$$\omega_k(f; 1/n) \leqq n^{-k} B_k \sum_{i=1}^{n} i^{k-1} A_i(f),$$

$$A_\sigma(f) \leqq B_k \omega_k(f; 1/\sigma).$$

R. P. Boas, Jr. (Evanston, Ill.).

Gopengauz, I. E.; and Timan, A. F. On the modulus of continuity of periodic functions with a given modulus of smoothness. Uspehi Mat. Nauk (N.S.) **12** (1957), no. 3(75), 291–294. (Russian)

The following theorem is proved. Let $f(x)$ be a measurable periodic function satisfying the inequality

$$(1)\qquad |f(x_1)+2f\left(\frac{x_1+x_2}{2}\right)+f(x_2)|\leq M|x_1-x_2|^{\alpha}\quad (0<\alpha\leq 1)$$

uniformly for all x_1, x_2 on the real axis, then

$$(2)\qquad \omega(f;h)\leq\begin{cases}\dfrac{4M}{3\ln 3}\,h\ln\dfrac{1}{h}+O(h), & \text{if } \alpha=1,\\[2ex] \dfrac{2^{\alpha+1}M}{3-3^{\alpha}}\,h^{\alpha}+O(h), & \text{if } 0<\alpha<1,\end{cases}$$

where $\omega(f;h)$ is the modulus of continuity, i.e.,

$$\omega(f;h)=\sup_{|x_1-x_2|\leq h}|f(x_1)-f(x_2)|.$$

The constants $4/(3\ln 3)$, and $2^{\alpha+1}/(3-3^{\alpha})$ in the inequality (2) are not the smallest possible ones. This work is an extension of the second author's earlier work [Dokl. Akad. Nauk SSSR (N.S.) **70** (1950), 961–963; MR **11**, 422].

H. P. Thielman (Ames, Iowa).

Brudnyĭ, Yu. A. On the maximum modulus of a quasi-smooth function. Uspehi Mat. Nauk (N.S.) **12** (1957), no. 4(76), 273–275. (Russian)

A function f is said to be quasi-smooth if $|\Delta_h^2 f(x)|\leq M2h$, where M is a constant and $x-h$, $x+h$ lie in the closed interval $[a, b]$, which is the domain of definition of $f(x)$. The constant K is defined as $K=\sup_{f\in A}\sup f(x)$, where A is the class of all continuous functions defined in $[-1, 1]$ and satisfying the conditions $|\Delta_h^2 f(x)|\leq 2h$, $f(-1)=f(1)=0$.

The author proves the following theorems. Let $f(x)\geq 0$. If $\max_{-1\leq x\leq 1} f(x)=f(x_0)=K-\varepsilon$ $(\varepsilon>0)$, and if $x_1\leq x_2\cdots\leq x_n$ are the roots of the equation $f(x)=t$, $t<K-\varepsilon$, and if furthermore $x_i<x_0<x_{i+1}$, then

$$K\leq 2(t+\varepsilon)/[2-(x_{i+1}-x_i)].$$

If $f\in A$, and $f(x)=f(-x)$ for $0\leq x\leq\frac{1}{2}$, then

$$\max_{-1\leq x\leq 1}|f(x)|\leq 5/4.$$

This estimate is exact.

The problem was proposed by A. F. Timan [Izv. Akad. Nauk. SSSR Ser. Mat. **15** (1951), 243–254; MR **13**, 17].

H. P. Thielman (Ames, Iowa).

Ćetković, Simon. Formation de certaines fonctions réelles d'une multitude finie des variables aux propriétés intéressantes. Bull. Soc. Math. Phys. Serbie **8** (1956), 169–182. (Serbo-Croatian. French summary)

The author constructs certain functions of n real variables illustrating the independence of continuity, discontinuity, existence of partial derivatives and differentiability.

Guglielmino, Francesco. Un criterio di compattezza rispetto alla convergenza quasi uniforme del tipo semiregolare. Ricerche Mat. **4** (1955), 150–159.

A sequence $\{f_n\}$ of functions defined on the unit square K is said to converge quasi-uniformly in a semiregular way with respect to y, if for any ε it converges uniformly except in a set whose projection on the x-axis has measure less than ε. A family $\{f_n\}$ of functions defined in K is said to be equi-quasibounded in a semiregular way with respect to y if for any $\varepsilon>0$ there is an M_ε such that $|f(x, y)|\leq M_\varepsilon$ except on a set whose projection on the x-axis has measure less than ε. The author concerns himself with a class S of functions f defined in K, which are measurable with respect to x for almost all y and absolutely continuous in y for almost all x, and for each of which the integral $\int_y^{y+1/(m+1)} f(y, t)dt$ is of bounded variation in the rectangle $[0, 1; 0, m/(m+1)]$. Denote by $V_f^m(y)$ the total variation with respect to x of that integral. The author's theorem is as follows. Let $\{f_n\}$ be a sequence of functions of the class S, equi-quasi bounded in a semiregular way in K with respect to y and such that: (i) $\iint_K |\partial f_n/\partial y|^{1+\alpha}dxdy<A$, (ii) $\int_0^{m/(m+1)} V_{f_n}^m(y)dy<A$, for $m, n=1, 2, \cdots$, where A and α are positive constants. Then any subsequence of $\{f_n\}$ contains a further subsequence, converging quasi-uniformly in a semi-regular way with respect to y, whose limit is a function of the class S.

J. M. Danskin (Princeton, N.J.).

Flett, T. M. Some remarks on a maximal theorem of Hardy and Littlewood. Quart. J. Math., Oxford Ser. (2) **6** (1955), 275-282.

Es sei $f(x)\geq 0$ in $a\leq x\leq b$ integrierbar und $\Theta(x)=\sup_{a\leq t<x}((x-\xi)^{-1}\int_\xi^x f(t)dt)$. Hardy und Littlewood [Acta Math. **54** (1930), 81–116] haben bewiesen, daß für $p>1$

$$(1)\qquad \int_a^b \Theta^p(x)dx\leq\left(\frac{p}{p-1}\right)^p\int_a^b f^p(x)dx,$$

für $p=1$

$$(2)\qquad \int_a^b \Theta(x)dx\leq B_1(a, b)\int_a^b f(x)\log^+ f(x)dx+B_2(a, b)$$

gilt. Verf. gibt, mit Benützung einiger Einzelheiten der Beweise von F. Riesz [J. London Math. Soc. **7** (1932), 10–13] und von Wiener [Duke Math. J. **5** (1939), 1–18], einen neuen Beweis für diese Ungleichungen und zeigt, daß man in (2), bei beliebigem $0<k<1$, $B_1(a, b)=1/(1-k)$, $B_2(a, b)=(b-a)/k$ wählen kann. Ebenfalls mit Hilfe der Methode von Wiener und durch Anwendung von (2) beweist er folgende Verallgemeinerung eines Satzes von Burkill [J. London Math. Soc. **26** (1951), 244–249; MR **13**, 332]: es sei $f(x, y)=f(P)\geq 0$ meßbar in

$$S=\{(x, y): 0\leq x\leq 1, 0\leq y\leq 1\},$$

$\int_S f(P)\log^+ f(P)dP<+\infty$ und, für $P_0\in S$, $f^*(P_0)=\sup |I|^{-1}\int_I f(P)dP$, wo I die P_0 enthaltenden achsenparallelen Rechtecke ICS durchläuft; dann hat man für $\alpha>0$

$$|\{P: f^*(P)>\alpha\}|\leq\frac{16}{\alpha}\int_{f\geq\alpha/4} f(P)\left[1+\log\left(\frac{4f(P)}{\alpha}\right)\right]dP.$$

Bei der Korrektur bemerkt er, daß im wesentlichen derselbe Beweis von (1) (unpubliziert) auch von Stein gefunden wurde.

A. Császár (Zbl **67** (1957), 39).

Dvoretzky, Aryeh. Remark on my paper "On a theorem of J. L. Walsh". Proc. Amer. Math. Soc. **8** (1957), 982.

The author corrects the enunciation of a theorem which is given in an earlier paper by him [same Proc. **7** (1956), 363–366; MR **18**, 116]. The amendment was quoted in the review of the earlier paper.

U. S. Haslam-Jones.

MacLane, G. R. Continuity properties of derivatives of sequences of functions. Proc. Amer. Math. Soc. **8** (1957), 897–898.

The author gives the counter-example to the original form of a theorem of A. Dvoretzky [see preceding review] which led to its correction.

U. S. Haslam-Jones.

Lipiński, J. S. Sur certains problèmes de Choquet et de Zahorski concernant les fonctions dérivées. Fund. Math. **44** (1957), 94–102.

G. Choquet [J. Math. Pures Appl. (9) **26** (1947), 115–226; MR **9**, 419] proved that if $f(x)$ has a bounded derivative $f'(x)$ and E is the set of points at which $f'(x)=a$ for some constant a, then E is a set G_δ with a certain metric density property. He asked whether this necessary condition was also sufficient. Z. Zahorski [Trans. Amer. Math. Soc. **69** (1950), 1–54; MR **12**, 247] found a necessary and sufficient condition M_4 that E may be the set of points at which $f'(x)>a$ for some constant a, and asked a question concerning a possible alternative formulation of the condition M_4. The present author describes the construction of a set which answers both these questions in the negative. *U. S. Haslam-Jones* (Oxford).

Tzodiks, V. M. On sets of points where the derivative is finite or infinite correspondingly. Dokl. Akad. Nauk SSSR (N.S.) **114** (1957), 1174–1176. (Russian)

The writer completes results of Lusin, Zahorski, Brudno, and Landis [Landis, same Dokl. (N.S.) **107** (1956), 202–204; MR **17**, 1190] by proving the following theorem. Let E be an $F_{\sigma\delta}$ of measure zero, and let N be a G_δ containing E and contained in the set of real numbers; then there is a continuous increasing function F such that $F'(x)=+\infty$ if $x\in E$, the lower derivative of F is finite at x if $x\notin E$, and the derivative $F'(x)$ exists and is finite if $x\notin N$. *M. M. Day* (Urbana, Ill.).

Tzodiks, V. M. On sets of points where the derivative is $+\infty$ or $-\infty$ correspondingly. Dokl. Akad. Nauk SSSR (N.S.) **113** (1957), 36–38. (Russian)

Continuing work of the note reviewed above it is proved that: In order that two sets of real numbers E_1 and E_2 be the sets where some everywhere-finite real-valued function has derivative $+\infty$ and $-\infty$, respectively, it is necessary and sufficient that (1) E_1 and E_2 be $F_{\sigma\delta}$ sets of measure zero, and (2) there exist two disjoint F_σ sets H_1 and H_2 such that $H_1\supset E_1$ and $H_2\supset E_2$. *M. M. Day* (Urbana, Ill.).

Measure, Integration

★ **Wernikoff, Robert E. Outline of Lebesgue theory: A heuristic introduction.** Tech. Rep. 310. Research Laboratory of Electronics, Massachusetts Institute of Technology, Cambridge, Mass. v+74 pp.

The purpose of this outline is to acquaint communications engineers with the language in which much of the mathematical foundation of communication theory is most frequently and most naturally expressed. *From the preface.*

Erdös, P.; and Kakutani, S. On a perfect set. Colloq. Math. **4** (1957), 195–196.

This article is taken from a letter of P. Erdös to E. Marczewski, one paragraph of which is as follows.

···Enclosed I send you our promised solution to your problem **P** 125, Colloq. Math. **3** (1954), p. 75. The problem is this: A linear set S is said to have property (S_n) if there exists an η_n such that if $x_1<x_2<\cdots<x_n$ with $x_n-x_1<\eta_n$ are any n real numbers, there exist n elements $y_1, y_2, \cdots, y_n$ of S congruent to $x_1, x_2, \cdots, x_n$. You ask: Does there exist a perfect set S of measure 0 having property (S_n)? Kakutani and I have constructed a perfect set S of measure 0 having property (S_n) for all $n\geqq 2$. Our set S is defined as the set of non-negative numbers

$$\sum_{k=2}^{\infty} a_k/k!, \quad 0\leqq a_k\leqq k-2.$$

Pucci, Carlo. A proposito di un teorema riguardante la misura di involucri di insiemi. Boll. Un. Mat. Ital. (3) **12** (1957), 420–421.

Correction to the proof of the theorem in Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) **20** (1956), 294–298 [MR **18**, 567] along lines indicated by M. Kneser.

Tortrat, Albert. Sur le produit de composition des mesures singulières. C. R. Acad. Sci. Paris **244** (1957), 1446–1448.

Nous donnons des critères permettant d'affirmer la singularité ou l'absolue continuité de certaines classes de mesures. Nous en déduisons des exemples de mesures singulières dont le produit de composition est également singulier. *Résumé de l'auteur.*

Nikodým, Otton Martin. On extension of a given finitely additive field-valued, non negative measure, on a finitely additive Boolean tribe, to another tribe more ample. Rend. Sem. Mat. Univ. Padova **26** (1956), 232-327.

Kluvánek, Igor. Abstract integral as a positive functional and the theorem on extension of measure. Mat.-Fyz. Časopis. Slovensk. Akad. Vied **6** (1956), 3–9. (Czech. Russian summary)

The author proves two lemmas about integrals on an abstract space which lead to the well-known theorem: if μ is a σ-finite measure on a ring R, then there exists a unique complete measure $\bar{\mu}$ on a certain σ-ring S containing R such that $\mu(E)=\bar{\mu}(E)$ for sets E in R. *From the author's summary.*

Mařík, Jan. The surface integral. Czechoslovak Math. J. **6(81)** (1956), 522–558. (Russian summary)

Let E_m be the Euclidean m space and $|x|$ the Euclidean norm. The norm $\|A\|$ of a measurable bounded set $A\subset E_m$ is defined as the supremum of the integral $\int_A \operatorname{div} v\,dx$, where the supremum is taken in the class $\mathfrak{B}_A$ of all real vectors $v=v(x)=(v_1, \cdots, v_m)$, $x\in E_m$, where each v_j is a polynomial in $x_1, \cdots, x_m$ and $|v|\leqq 1$ in A. The class $\mathfrak{A}$ of all measurable bounded sets $A\subset E_m$ is considered for which $\|A\|<+\infty$. The class $\mathfrak{A}$ is very large and admits of various characterizations. Let E_{m-1}^j denote the $m-1$ space of points $y^j=(x_1, \cdots, x_{j-1}, x_{j+1}, \cdots, x_m)$ for each $j=1, \cdots, m$, and let $A(y^j)$ denote the intersection of A with the straight line $x^j=y^j$. If $A(y^j)$ differs from the union of $r=r(y^j)<+\infty$ disjoint intervals $(a_1, b_1), \cdots, (a_r, b_r)$ for a set of linear measure zero, and if (a) the function $r=r(y^j)$ is B-measurable and L-integrable in E_{m-1}^j, (b) the sets A_-, A_+ of all points $(y^j, a_i), (y^j, b_i)$, $i=1, \cdots, r$, are B-measurable in E_m ($j=1, \cdots, m$), then $A\in\mathfrak{A}$. In particular any convex bounded set belongs to $\mathfrak{A}$. The author shows that for every set $A\in\mathfrak{A}$ with (compact) boundary D, there exists a measure p on D (surface measure) and a vector ν on D with $|\nu|=1$ (normal vector) such that

$$P(A, v)=\int_A \operatorname{div} v\,dx=\int_D (v\cdot\nu)\,dp,\ p(D)=\|A\|,$$

for all vectors $v\in\mathfrak{B}_A$, where $(v\cdot\nu)$ denotes the scalar product. Thus P depends only on the values of v on the

boundary D of A as in the elementary case. By a limit process $P(A, v)$ is then extended into a linear functional $R(A, f)$ defined for all bounded vectors $f(x)$, $x \in D$, B-measurable on D, where A is any set $A \in \mathscr{A}$. Also the relation holds

$$R(A, f) = \int_D (f \cdot v) d\mu.$$

If $T = (\varphi_1, \cdots, \varphi_m)$ is any continuous one-one mapping of class C_1 from an open set $G \subset E_m$ into the set $T(G) \subset E_m$ with functional determinant different from zero everywhere in G, the functional matrix $M(x)$ is defined as the matrix whose rows are grad φ_1, $\cdots$, grad φ_m. Then it is proved that for every set $A \in \mathscr{A}$, $\bar{A} \subset G$, with boundary D, for every bounded Borel vector $w(x)$, $x \in D$, we have $R(T(A), w) = R(A, v)$, where $v(x) = \text{adj } M(x) \cdot w(T(x)) \cdot \text{sgn det } M(x)$. [Previous papers of the author on the subject: Časopis Pěst. Mat. **79** (1954), 3–40; **81** (1956), 79–82; Czechoslovak Math. J. **5(80)** (1955), 467–487; **6(81)** (1956), 387–400; MR **16**, 492; **19**, 256; **18**, 796. For the literature and relevant work on the subject (not quoted) see K. Krickeberg, Math. Nachr. **10** (1953), 261–314; **11** (1954), 35–60; **12** (1954), 341–365; MR **15**, 611, 692; **16**, 807.]

L. Cesari (Lafayette, Ind.).

Cecconi, Jaurès. La disuguaglianza di Cavalieri per la k-area secondo Lebesgue in un n-spazio. Ann. Mat. Pura Appl. (4) **42** (1956), 189–204.

A Cavalieri-type inequality between the Lebesgue area of a continuous parametric surface and the lengths of its "sections" has been given by Cesari [Surface area, Princeton, 1956, Ch. VI; MR **17**, 596], and used in the direct proof of the identities of Peano, Geocze and Lebesgue areas, as well as in the calculus of variations [Cesari, Amer. J. Math. **74** (1952), 265–295; MR **14**, 292; Cecconi, MR **18**, 645; see also MR **19**, 751]. In the same line the author proves an analogous inequality between the k-dim. Lebesgue area $L_k(S)$ of a continuous k-dim. parametric variety $S: p = p(w)$, $p = (x_1, \cdots, x_n) \in E_n$, $w = (u_1, \cdots, u_k) \in A$, $A = [|u_i| \leq 1, i = 1, \cdots, k]$, and the $(k-1)$-dim. areas of the "sections" $S(t)$ of S, say $\alpha_{k-1}[S(t)]$:

$$HL_k(S) \geq \int_{-\infty}^{+\infty} \alpha_{k-1}[S(t)] dt.$$

The sections $S(t)$ of S are defined as by Cesari as follows: Let $f(p)$, $p \in E_n$, be a Lipschitz function of p in E_n of coefficient H, and for every real t let $D^-(t)$ [$D^+(t)$] be the subset of A, open in A, where $f[p(w)] < t$ [$> t$]. Then the section $S(t)$ of S is the boundary of $D^-(t)$ in A [or of $D^+(t)$]. For each t, the $(k-1)$-dim. area $\alpha_{k-1}[S(t)]$ is defined by considering sequences $[P_n]$ of polyhedral varieties converging uniformly toward S in A, and sequences $[F_n]$ of figures $F_n \subset D^-(t)$, invading $D^-(t)$. Then $\alpha_{k-1}[S(t)]$ is conveniently defined as the lower limit of the $(k-1)$-dim. areas of the restrictions of the P_n over the boundaries in A of the F_n. For $k=2$ the last definition is proved to be equivalent to the one of Cesari. A slight oversight of the author is being corrected by an erratum in process of publication in the same journal.

The invariance of the $(k-1)$-area with respect to Fréchet and Lebesgue equivalence is also discussed {F- and L-equivalences have to be understood as defined with respect to the (restricted) class of all homeomorphisms which can be approximated by means of quasi-linear homeomorphisms}.

L. Cesari (Lafayette, Ind.).

★ **Kampé de Fériet, J. Séminaire sur les problèmes mathématiques de la physique théorique. VIII. Construction de mesures dans certains espaces fonctionnels en vue des applications à la physique mathématique.** Université de Lille, 1956–1957. 22 pp.

Article de séminaire exposant des résultats connus. L'auteur donne un moyen simple de construire une loi de probabilité (mesure ≥ 0 de masse 1) sur un espace de Hilbert séparable H. Le procédé consiste toujours à plonger H dans un compact K, H étant muni d'une topologie plus fine que la topologie induite par K, et à construire une loi de probabilité de Radon sur K, pour laquelle H soit de mesure 1. Soit par exemple τ une loi de probabilité de Radon sur la droite réelle R, telle que $\int x\, d\tau = 0$, $\int x^2 d\tau = 1$, $\int x^4 d\tau = k < \infty$. Soit $(\xi_n)_{n=1,2,\ldots}$ une suite de variables aléatoires réelles indépendantes de même loi τ; alors $\xi = (\xi_1, \xi_2, \cdots, \xi_n, \cdots)$ est une variable aléatoire sur R^∞ donc aussi sur le compact $K = \bar{R}^\infty$, $\bar{R}$ étant la droite achevée. Soit $(\lambda_n)_{n=1,2,\ldots}$ une suite de nombres réels >0, de somme $\sum \lambda_n$ finie. L'espace de Hilbert H des suites réelles $X = (X_n)_{n=1,2,\ldots}$ pour lesquelles $\sum \lambda_n X_n^2 < \infty$ est plongé dans l'espace R^∞ de toutes les suites réelles donc dans K. D'après le théorème de Kolmogorov la série aléatoire $\sum \lambda_n \xi_n^2$ est presque sûrement convergente, donc ξ est presque sûrement dans H, et définit ainsi une loi de probabilité sur H.

L. Schwartz (Paris).

See also: Topological Groups: Mackey.

Functions of Complex Variables

★ **Gigueaux, M.; et Oudart, A. Aide-mémoire des fonctions analytiques.** Libraire Polytechnique Ch. Béranger, Paris-Liége, 1958. i+35 pp. 700 francs.

A set of standard theorems and formulas, with occasional comments, under the ten headings: généralités, théorème de Cauchy, formule de Cauchy, série de Laurent, fonction analytique en un point, intégrale le long d'un contour fermé, fonction multiforme à dérivée holomorphe, représentation conforme, représentation conforme d'un domaine borné par une courbe fermée, quelques transformations conformes.

★ **Dienes, P. The Taylor series: an introduction to the theory of functions of a complex variable.** Dover Publications, Inc., New York, 1957. x+552 pp. $2.75.

An unchanged reprint of the book first published by the Clarendon Press, Oxford, in 1931.

★ **Golusin, G. M. Geometrische Funktionentheorie.** Hochschulbücher für Mathematik, Bd. 31. VEB Deutscher Verlag der Wissenschaften, Berlin, 1957. xii+438 pp. DM 39.60.

A translation by Doris-Ursula Pirl under the editorship of Udo Pirl from the Russian of the book reviewed in MR **15**, 112.

Dalovitch, Voĭn. Sur une propriété de la résultante des deux fonctions. Bull. Soc. Math. Phys. Serbie **8** (1956), 165–168.

On démontre le théorème suivant: si le cercle de convergence de la série $\sum a_\nu z^\nu$ est le cercle-unité, on peut pour une série quelconque $\sum c_\nu z^\nu$ convergente dans le cercle-unité déterminer une série majorante $\sum b_\nu z^\nu$ telle que la résultante $\sum a_\nu b_\nu z^\nu$ soit une fonction entière de $1/(1-z)$.

Du résumé de l'auteur.

Noble, M. E. A converse theorem on overconvergence of sequences of partial sums. Proc. Cambridge Philos. Soc. **53** (1957), 592–598.

Let the power series $\sum a_n z^n$ have gaps: $a_n=0$ for $n_k<n\leqq N_k$; let the function f which it represents be holomorphic on the open arc J of the unit circle; let j be a closed subarc of J, and let $\phi(n)$ denote the minimal convex majorant of $\log(2+|a_n|)$. It is known that there exist constants $h(J, j)$ and $c(J, j)$ $(c<1)$, independent of f, such that the relation

$$\liminf (N_k-n_k)/\phi(n_k)\geqq h$$

implies that $s_{n_k}(z)-f(z)=O(c^{N_k-n_k})$, uniformly on j. The author establishes a converse theorem. For the special case where $\{a_n\}$ is bounded, it takes the following simple form. If $f(z)$ is continuous on the union of the unit disk and the arc J on the unit circle, and if $s_{n_k}(z)\to f(z)$ uniformly on some subarc of J, then there exists a sequence $\{\Delta_k\}(\Delta_k\to\infty)$ such that

$$\lim_{k\to\infty} \max_{n_k\leq m\leq n_k+\Delta_k} |a_m|=0.$$

G. Piranian (Ann Arbor, Mich.).

Uchiyama, Saburô. Complex numbers with vanishing power sums. Proc. Japan Acad. **33** (1957), 10–12.

Recently, Sós and Turán [Acta Math. Acad. Sci. Hungar. **6** (1955), 241–255; MR **17**, 1061] were led to the problem of determining the set $\mathfrak{Z}_{m,n}$ of systems of n complex numbers $z_1, z_2, \cdots, z_n$ for which $s_\nu=\sum_{j=1}^n z_j^\nu=0$ $(\nu=m+1, \cdots, m+n-1)$ (m a non-negative integer). In the present paper, for general m and n, a characterization is given of the systems in $\mathfrak{Z}_{m,n}$. *C. G. Lekkerkerker.*

Raljević, Š. Sur certaines classes de polynômes et sur la répartition de leurs zéros. Srpska Akad. Nauka. Zb. Rad. **50**, Mat. Inst. 5 (1956), 1–60. (Serbo-Croatian. French summary)

L'objet de ce travail est l'interprétation géométrique des transformations linéaires (affines) effectuées dans les polynômes de la forme

$$(*)\qquad \sum_{k=0}^{\nu} \frac{n}{n-k}\binom{n-k}{k}(-p^2)^k z^{n-2k}+q,\ \nu=[n/2],$$

et en particulier la répartition des zéros des polynômes transformés ainsi obtenus. Par des considérations géométriques élémentaires, l'auteur établit que la construction des zéros des polynômes transformés se déduit avec la même précision que celle des polynômes réguliers correspondants aux (*). La démonstration d'une suite de résultats de ce genre repose sur la notion et les propriétés de "l'écart" d'un polynôme $P_n(x)$ défini par

$$R=\max|P_n(\zeta_\nu)|-\min|P_n(\zeta_\nu)|,\ P_n'(\zeta_\nu)=0$$
$$(\nu=1, 2, \cdots, n-1).$$

M. Tomić (Beograd).

Dobrzycki, S. On the geometry of the zeros of polynomials. Prace Mat. **2** (1956), 94–116. (Polish. Russian and English summaries)

The paper gives information on a few basic theorems concerning the relative positions of zeros of a polynomial and its derivative, or some other particular polynomials, and the determination, according to the coefficients of the polynomial, of the regions containing all its zeros or only a given number of them. *Author's summary.*

Rao, K. Narasimha Murthy. Two theorems on bounded functions. J. London Math. Soc. **32** (1957), 430–435.

Functions $f(z)$ regular and satisfying $|f(z)|<1$ in $|z|<1$ are called bounded by the author. He obtains for such functions the exact upper bound for $f'(z)$ at an arbitrary point z in $|z|<1$ when either (a) $f(0)=k$ is given; or (b) f has a zero of order n at the origin. This extends a classical result of Dieudonné [Ann. Sci. Ecole Nom. Sup. (3) **48** (1931), 247–358], who considered the case when $k=0$ in (a) or equivalently $n=1$ in (b). *W. K. Hayman* (London).

Biernacki, Mieczysław; and Krzyż, Jan. On the monotonity of certain functionals in the theory of analytic functions. Ann. Univ. Mariae Curie-Skłodowska. Sect. A. **9** (1955), 135–147 (1957). (Polish and Russian summaries)

For a function $w=f(z)$ regular in $|z|\leqq r$ let $L(r)$ be the length of the image of $|z|=r$, $S(r)$ the area of the image of $|z|<r$ with due count of multiplicity. The authors prove that $L^2(r)-4\pi S(r)$ increases strictly with r, unless $f(z)$ is bilinear. Further if $I_2(r, f)=(2\pi)^{-1}\int_0^{2\pi}|f(re^{i\theta})|^2 d\theta$, they show that $r^2 I_2(r, f')/I_2(r, f)$ increases strictly with r unless $f(z)$ reduces to az^n. They also conjecture that $L^2(r)/S(r)$ increases with r. *W. K. Hayman* (London).

Uluçay, Cengiz. Note on a generalization of Schwartz' lemma. Comm. Fac. Sci. Univ. Ankara. Ser. A. **8** (1956), 1–5. (Turkish summary)

According to an earlier theorem of the reviewer [Duke Math. J., **14** (1947), 1035–1049; MR **9**, 340], the Schwarz lemma for bounded analytic functions $w=f(z)$ will remain true even if $f(z)$ is not single-valued in $|z|<1$, provided $f'(z)$ remains finite. In the present paper, the author shows that the result remains valid if the boundedness of $f(z)$ in $|z|<1$ is replaced by the somewhat weaker assumption that a suitably defined Riemannian metric on the Riemann surface over the w-plane associated with the mapping $z\to w$ should be of Gaussian curvature $\leqq -4$.

Z. Nehari (Pittsburgh, Pa.).

Uluçay, C. Bloch functions of the third kind and the constant $\mathfrak{A}$. Proc. Amer. Math. Soc. **8** (1957), 923–925.

Let A_f be the sup of radii of circles contained in $f(|z|<1)$, and S the class of functions $f(z)=z+a_2z^2+\cdots$, schlicht and analytic for $|z|<1$. Then the number, $\inf_{f\in S} A_f$, is the Bloch-Landau constant $\mathfrak{A}$. This reviewer [same Proc. **7** (1956), 75–76; MR **17**, 1066] showed that $\mathfrak{A}>.569$, which is but a tiny improvement of Landau's original bound. The author now attempts to show that $\mathfrak{A}>.629$, but, unfortunately, his reasoning is completely invalid. If the line of proof were correct, it would actually show that if $f\in S$, and $a_2=0$, then f omits no values with magnitude $<.629$; a counterexample is $z/(1+z^2)$. The specific cause of error in the proof is the failure to consider the restrictions on t necessary to insure that the intermediate function $\tilde{f}$ is schlicht. *E. Reich.*

★**Sario, Leo. Functionals on Riemann surfaces.** Lectures on functions of a complex variable, pp. 245–256. The University of Michigan Press, Ann Arbor, 1955. $10.00.

Antoniuk, G. K. Covering of areas for functions regular within a ring. Dokl. Akad. Nauk SSSR (N.S.) **114** (1957), 9–10. (Russian)

Let $\mathfrak{M}$ denote the class of functions $f(z)$ regular in the ring D_R: $1<|z|<R$ for which $\Delta \arg f(z)\geqq 2\pi$ on any con-

tour in D_R not homologous to zero. Let $\mathfrak{R}$ be the image of D_R under $f(z)$. Hažaliya [Akad. Nauk Gruzin. SSR. Trudy Mat. Inst. Razmadze **18** (1951), 245–256; MR **14**, 549] attempted to define the star of $\mathfrak{R}$. His definition was not clear, and the present author undertakes to make this concept precise. He then announces that if $f(z) \in \mathfrak{M}$ then $(1+S/\pi)(1+s/\pi) \geqq R^4$, where S is the area of the star of $\mathfrak{R}$ and s is the area of its prei-mage. However, some normalizing condition must be imposed on the family $\mathfrak{M}$ for this theorem to be correct. *A. W. Goodman.*

Royden, H. L. Rings of analytic and meromorphic functions. Trans. Amer. Math. Soc. **83** (1956), 269–276.

Let $A(W)$ denote the ring of all analytic functions on an open Riemann surface W. The paper begins with a discussion of the ideal theory of $A(W)$; in particular, the principal maximal ideals and the maximal principal ideals are characterized. Florack's generalization [Schr. Math. Inst. Univ. Münster no. 1 (1948); MR **12**, 251] of the Weierstrass theorem to open Riemann surfaces is utilized here. The principal result is the following. Th. 1: If W_1 and W_2 are open Riemann surfaces and if ϕ is a homomorphism of $A(W_2)$ into $A(W_1)$ which preserves the constants, then there is a unique analytic mapping ψ of W_1 into W_2 such that $\phi(f)=f\circ\psi$. In particular, if ϕ is an isomorphism onto, then W_1 and W_2 are conformally equivalent. Th. 2 shows that the conformal type of certain planar domains D can be characterized by the algebraic structure of rings R of meromorphic functions, if R contains all bounded analytic functions on D. An example shows that this technique cannot be extended to all nonplanar domains. There are also some propositions concerning valuations on fields of meromorphic functions. *W. Rudin* (Rochester, N.Y.).

Ohtsuka, Makoto. Sur les ensembles d'accumulation relatifs à des transformations localement pseudo-analytiques au sens de Pfluger-Ahlfors. Nagoya Math. J. **11** (1957), 131–144.

The object of this paper is to improve the author's previous results [Ann. Inst. Fourier, Grenoble **5** (1953–1954), 29–37; MR **17**, 1191] in the following points. In the present paper, the author considers a space $\mathfrak{F}$ composed of abstract Riemann surfaces, defines a boundary element of $\mathfrak{F}$ by means of a filter of a very general character and makes use of a transformation locally pseudo-analytic in the sense of Pfluger-Ahlfors with the exception of a certain closed set. This paper is perhaps the first contribution to the theory of cluster sets which is made concerning mappings of abstract Riemann surfaces into abstract Riemann surfaces. *K. Noshiro* (Nagoya).

Ibragimov, I. I. Extremal problems in a class of entire functions of exponential type. Uspehi Mat. Nauk (N.S.) **12** (1957), no. 3(75), 323–328. (Russian)

The class (w_σ) consists of those functions of exponential type σ which belong to L^2 on the real axis and hence, by the Paley-Wiener theorem, are of the form $f(z)=(2\pi)^{-\frac{1}{2}}\int_{-\sigma}^{\sigma} e^{iuz}\phi(u)du$, $\phi \in L^2$. Consider operators of the form $J(f)=(2\pi i)^{-1}\int_{-\sigma}^{\sigma}\phi(u)\Phi(iu)du$ (the author considers only the special case when $\Phi(z)$ is again an entire function of exponential type). We have, by Schwarz's inequality and Plancherel's theorem, $|J(f)|\leq(2\pi)^{-\frac{1}{2}}\|f\|(\int|\Phi|^2du)^{\frac{1}{2}}$, where the norm is the L^2 norm on $(-\infty,\infty)$. Hence, in particular, $|f^{(n)}(x)|\leq(\sigma/\pi)^{\frac{1}{2}}\sigma^n(2n+1)^{-\frac{1}{2}}\|f\|$ [the case $n=1$ was obtained by Korevaar, Nieuw Arch. Wisk. (2) **23** (1949), 55–62; MR **10**, 289]. Thus the author has an inequality for the coefficients a_k in the Maclaurin series of an element of w_σ. He deduces that for any C_k we have

$$\left|\sum_{k=0}^{n} a_kC_k\right| \leq (\sigma/\pi)^{\frac{1}{2}}\|f\| \sum_{k=0}^{n}\frac{|C_k|\sigma^k}{k!(2k+1)^{\frac{1}{2}}}.$$

In particular, for any real α and β $(\beta>\alpha)$

$$\left|\int_\alpha^\beta f(x)dx\right| \leq (\sigma/\pi)^{\frac{1}{2}}\|f\| \sum_{k=0}^{\infty}\frac{\sigma^k(\beta^{k+1}-\alpha^{k+1})}{(k+1)!(2k+1)^{\frac{1}{2}}}.$$

Another application of the original inequality is

$$|f(\beta)-f(\alpha)| \leq (2\sigma/\pi)^{\frac{1}{2}}\|f\|\left\{1-\frac{\sin(\beta-\alpha)\sigma}{(\beta-\alpha)\sigma}\right\}.$$

R. P. Boas, Jr. (Evanston, Ill.).

Shah, S. M. Exceptional values of entire and meromorphic functions. II. J. Indian Math. Soc. (N.S.) **20** (1956), 315–327.

The author continues part I [Duke Math. J. **19** (1952), 585–593; MR **14**, 365]. He proves a large number of results connecting the characteristic function $T(r)$ with the numbers $n_1(r, a)$ of simple a-points, $n_2(r, a)$ of simple and double a-points (each counted once), and $\bar{n}(r, a)$ of a-points (each counted once), of a non-constant meromorphic or entire function F. The results take three pages to state; we quote some representative ones. In each case the stated number of possible exceptions can be attained. If F is meromorphic and of finite positive order ρ then $\liminf T(r)/n_1(r, x) \leq 10/\rho$ with at most 4 exceptions; if $\sum\delta(\alpha)>0$ then $10/\rho$ can be replaced by $4\rho^{-1}/\sum\delta(\alpha)$ with at most 3 exceptions. If F is meromorphic of finite order, and $\sum\delta(\alpha)>0$, then

$$\liminf T(r)/\{n_1(r, \alpha)\log r\} \leq 4/\sum\delta(\alpha)$$

with at most 3 exceptions. For any meromorphic F, $\liminf T(r)/\{n_1(r, x)\log r\} \leq 10$ with at most 4 exceptions. If a_1 and a_2 are E-exceptional then ρ is an integer and $n_1(r, x) \sim \rho T(r) \sim Cr^\rho$. If a_1 and a_2 are Borel exceptional then they are E-exceptional, even for F of infinite order. *R. P. Boas, Jr.* (Evanston, Ill.).

Falgas, Maurice. Sur les domaines étoilés vérifiant certaines propriétés liées à la propriété de convexité. C. R. Acad. Sci. Paris **244** (1957), 2275–2278.

La considération des transformées de Laplace-Borel des fonctions entières conduit à l'étude de domaines ayant la propriété A, c'est à dire admettant l'origine pour point intérieur, et réunions de domaines dont l'équation en coordonnées polaires r et φ est

$$r^\rho \leq r_0^\rho \cos[\rho(\varphi-\varphi_0)],$$

$|\varphi-\varphi_0| \leq \pi/2\rho$. L'auteur donne deux conditions nécessaires et suffisantes pour qu'un domaine étoilé par rapport à O ait la propriété A, et énonce le résultat suivant: si D est étoilé et a la propriété A, et si $z\to g(z)$ est une application conforme de $|z|<R$ sur l'extérieur de D, avec $g(0)=\infty$, l'extérieur de l'image de $|z|<R'<R$ est un domaine ayant la propriété A. *J. P. Kahane.*

Goldberg, A. A. An estimate of the sum of the defects of a meromorphic function of order less than unity. Dokl. Akad. Nauk SSSR (N.S.) **114** (1957), 245-248. (Russian)

Shah [Math. Student **12** (1945), 67–70; MR **6**, 263] showed that if $a\neq b$ then $\delta(a)+\delta(b)\leq 1+\rho$ for a mero-

morphic function of order ρ $(0<\rho<1)$. The author shows that $\delta(a)+\delta(b)\leqq 1$ for $0\leqq\rho\leqq\frac{1}{2}$, while

$$\delta(a)+\delta(b)\leqq 2-2\rho\pi^{\frac{1}{2}}\Gamma(1-\tfrac{1}{2}\rho)/\Gamma(\tfrac{1}{2}-\tfrac{1}{2}\rho)$$

for $\frac{1}{2}<\rho<1$. The second bound is presumably not best possible. *R. P. Boas, Jr.* (Evanston, Ill.).

Kung Sun. Contributions to the theory of schlicht functions. I. Distortion theorems. Sci. Sinica **4** (1955), 229–249.

A translation into English of results already published in Chinese [Acta Math. Sinica **3** (1953), 231–250, 251–260; MR **17**, 142]. *A. W. Goodman* (Lexington, Ky.).

Shlionsky, H. G. On extremal problems for differentiable functionals in the theory of univalent functions. Dokl. Akad. Nauk SSSR (N.S.) **113** (1957), 280–282. (Russian)

The author considers certain classes of weakly differentiable functionals for functions meromorphic in a multiply connected region G. Then descending to the unit circle and the family S of functions $f(z)=z+\cdots$ regular and univalent there, he announces three theorems, each to the effect that if $f(z)$ is an extremal function in S for one of his functionals, then $f(z)$ satisfies a differential equation (too complicated to be reproduced here).

A. W. Goodman (Lexington, Ky.).

Lewandowski, Zdzisław. Quelques remarques sur les théorèmes de Schild relatifs à une classe de fonctions univalentes. Ann. Univ. Mariae Curie-Skłodowska. Sect. A. **9** (1955), 149–155 (1957). (Polish and Russian summaries)

Let Φ be the class of functions $f(z)=z-\sum_{n=2}^{\infty}a_nz^n$ regular in $|z|<1$ and such that $\sum_2^{\infty}n|a_n|\leqq 1$. The author proves that $f(z)\in\Phi$ maps $|z|<\frac{1}{2}$ onto a convex domain and $|z|<1$ onto a domain starlike with respect to the origin. These results had previously been proved by Schild [Proc. Amer. Math. Soc. **5** (1953), 115–120; MR **15**, 694] when in addition $f(z)$ is a polynomial and the a_n are nonnegative. Other results of Schild extend without change of proof.

The author further proves the sharp inequality

$$\left|\frac{f''(z)}{f'(z)}\right|\leqq\frac{1}{1-|z|},$$

for $f(z)\in\Phi$, when the a_n are nonnegative, and says that this is due to J. Krzyż. *W. K. Hayman.*

Orts, J. M^a. Recurrent series of Tchebycheff polynomials. Mem. Real Acad. Ci. Art. Barcelona **32** (1956), 269–281. (Spanish)

The author is interested on the one hand in comparing the coefficients of the expansion of the same analytic function in a Maclaurin series and in a series of Chebyshev polynomials $T_n(z)$, and on the other hand in comparing the functions $\sum a_nz^n$, $\sum a_nT_n(z)$ with the same coefficients. When $\{a_n\}$ is periodic, the simple properties of the first series are translated into somewhat less simple properties of the second. *R. P. Boas, Jr.*

Geronimus, Ya. L. On the equiconvergence of the Fourier-Čebyšev and Maclaurin expansions of analytic functions of class H_2. Dokl. Akad. Nauk SSSR (N.S.) **113** (1957), 491–492. (Russian)

The Fourier-Čebyšev expansion is the expansion in terms of polynomials $P_n(z)$ which are orthonormal on the unit circumference with weight $p(\theta)$. The author lists a variety of conditions, bearing on $p(\theta)$ and on the magnitude of

$$\left|z^n\overline{P}_n(1/z)-\exp\left\{-\frac{1}{4\pi}\int_0^{2\pi}\frac{e^{i\theta}+z}{e^{i\theta}-z}\log p(\theta)d\theta\right\}\right|$$

under which the difference between the partial sums of the two expansions in question converges uniformly to zero on a given interval. *R. P. Boas, Jr.*

Kaz'min, Yu. A. On the spectrum of systems of the form $\{z^n+\lambda f_n(z)\}$. Uspehi Mat. Nauk (N.S.) **12** (1957), no. 3(75), 155–158. (Russian)

Expand the $f_n(z)$ of the title in power series, $f_n(z)=\sum_k a_{kn}z^k$; let $\Delta(\lambda)$ be the determinant of the (infinite) matrix $((\delta_{kn}-\lambda a_{kn}))$; set $q_k=\sup_n|a_{kn}|$, $p_n=\sup_k|a_{kn}|$. Let A_r be the space of functions analytic $|z|<r$. Th. 1. For a number $R>1$ the following two conditions are equivalent: (a) If $1<r\leqq R$, then all $f_n(z)$ are in A_r and the sequence $\{z^n+f_n(z)\}$ is a power-like basis for A_r; (b) (1) $\limsup_k q_k^{1/k}\leqq 1/R$ and (2) $\Delta(1)$ is an absolutely convergent determinant whose value is not zero. The spectrum of $\{z^n+f_n(z)\}$ is the set of zeros of $\Delta(\lambda)$. Th. 2. If the system satisfies (1) and $\limsup_n p_n^{1/n}\leqq 1/\rho$, where $\rho>1$, then $\Delta(\lambda)$ is an entire function of order zero. *M. M. Day.*

Evgrafov, M. A.; and Soloviev A. D. On a general criterion for a basis. Dokl. Akad. Nauk SSSR (N.S.) **113** (1957), 493–496. (Russian)

A system of functions $u_n(z)$, regular in a region G, is a basis in $G_1\subset G$ if every function regular in G_1 can be expanded there in a unique uniformly convergent series of the $u_n(z)$. Theorem: If $u_n(z)=z^n\exp(-\lambda_n(z))$ where $\lambda_n(z)$ are regular in $|z|<R$, $\lambda_n(z)-\lambda_{n-1}(z)=\sum_{k=1}^{\infty}\Delta_{nk}z^k$, $\Delta_n{}^0(r)=\sum_{k=1}^{\infty}|\Delta_{nk}|r^k$, $l_n(r)=\sum_{k=0}^{\infty}\Delta_k{}^0(r)$, then $\{u_n\}$ is a basis in $|z|<R$ if $l_n(r)/n\to 0$ for every $r<R$. If $l_n(r)/n\to l(r)$, then $\{u_n\}$ is a basis in a smaller disk. Several corollaries are given. *R. P. Boas, Jr.* (Evanston, Ill.).

Kozmanova, A. A. Pólya's theorem for integral functions of two complex variables. Dokl. Akad. Nauk SSSR (N.S.) **113** (1957), 1203–1205. (Russian)

The theorem in question is the one connecting the indicator diagram of a function of exponential type with the singularities of its Borel transform. If $F(p_1, p_3)=\sum_{n=0}^{\infty}\sum_{m=0}^{n}a_{nm}p_3{}^{n-m}p_1{}^m$ is an entire function of exponential type, the author introduces the isotropic vector $\mathbf{p}(p_1, p_2, p_3)=\mathbf{p}'+i\mathbf{p}''$, with $\sum p_k{}^2=0$ and $\mathbf{p}'$, $\mathbf{p}''$ real, and the entire function

$$F(\mathbf{p})=\sum_{n=0}^{\infty}\sum_{m=0}^{n}a_{nm}p_3{}^{n-m}(p_1+ip_2)^m+\sum_{n=0}^{\infty}\sum_{m=0}^{n}a_{nm}{}^*p_3{}^{n-m}(p_1-ip_2)^m.$$

With $F(\mathbf{p})$ she associates the harmonic function

$$f(x, y, z)=\sum_{n=0}^{\infty}\sum_{m=-n}^{n}(-1)^m(n-|m|)!\times a_{nm}r^{-n-1}P_n{}^{|m|}(\cos\theta)e^{im\phi},\quad a_{n(-m)}=a_{nm}{}^*,$$

which plays the role of the Borel transform. The indicator of $F(\mathbf{p})$ is $h(\phi_1, \theta, \phi_2)=\limsup_{\rho\to\infty}\rho^{-1}\log|F(\mathbf{p})|$, where $\rho=|\mathbf{p}'|=|\mathbf{p}''|$. Then h is the supporting function of the convex cover of the set of singular points of f. Moreover, F and f are connected by the formulas

$$2\pi f(x, y, z) = \iint_S F(\mathrm{p})\rho^{-1}e^{-(\mathrm{pr})}ds,$$

$$\mathrm{p}F(\mathrm{p}) = 4\pi \iint_\sigma [\mathrm{grad}\, f\ \mathrm{n}\ \mathrm{p}]e^{(\mathrm{pr})}d\sigma,$$

where σ surrounds the singularities of f; n is the exterior unit normal vector of σ; s is a surface normal to $\mathrm{p}'(\rho, \phi_1', \theta')$ with ϕ_1' and θ' arbitrary but fixed; $[abc] = -(bc)a + (ca)b - (ab)c$. {Since there are several inconsistencies in the formulas as printed in the paper the reviewer does not guarantee that they are quoted here in accordance with the author's intentions.}

R. P. Boas, Jr. (Evanston, Ill.).

Smith, R. C. T. Bounded groups of transformations. Proc. Amer. Math. Soc. **8** (1957), 831–833.

It is known that any bounded group of inner transformations $T(\alpha)$ (in k-dimensional complex space) is similar to the group of its linear parts $L(T(\alpha))$, that is, there exists a transformation S whose linear part is the identity such that $T(\alpha) = S^{-1}L(T(\alpha))S$ for all α [see, e.g., S. Bochner and W. T. Martin, Several complex variables, Princeton, 1948, p. 19; MR **10**, 366]. This and related theorems have been concerned with existence and not with uniqueness. The paper under review is concerned with the question of uniqueness of S. *W. T. Martin.*

See also: Functions of Real Variables: Timan and Timan. **Harmonic Functions, Convex Functions:** Reade. **Special Functions:** Eichler. **Trigonometric Series and Integrals:** Ul'yanov. **Complex Manifolds:** Rothstein.

Geometric Analysis

See: Linear Algebra: Valentiner. **Geometries, Euclidean and Other:** Janekoski. **Differential Geometry:** Santaló; Reichardt; Gołab; Chern; Segre; Liber.

Harmonic Functions, Convex Functions

★ **Sario, Leo. Positive harmonic functions.** Lectures on functions of a complex variable, pp. 257–263. The University of Michigan Press, Ann Arbor, 1955. $10.00.

Reade, Maxwell O. Analogue of a theorem of F. and M. Riesz for minimal surfaces. J. Math. Soc. Japan **8** (1956), 177–179.

The principle of the maximum in complex variable theory has been shown by T. Radó and the reviewer [Trans. Amer. Math. Soc. **35** (1933), 648–661] to hold quite generally for functions of class PL — that is, for nonnegative functions $p(z)$, $z = x + iy$, of which the logarithm is subharmonic — and therefore to be applicable to the theory of minimal surfaces in conformal representation. The author now applies this principle to obtain the result, previously given by A. F. Monna [Nederl. Akad. Wetensch., Proc. **45** (1942), 687–689; MR **5**, 241], that if $p(z)$ is of class PL and bounded in $|z| < 1$, and if the set E of values θ for which $\lim_{r\to 1} p(re^{i\theta}) = 0$ is of positive measure, then $p(z) \equiv 0$. *E. F. Beckenbach.*

See also: Trigonometric Series and Integrals: Ul'yanov. **Complex Manifolds:** Spencer.

Special Functions

González, Mario O. Theory of elliptic functions. Rev. Soc. Cubana Ci. Fís. Mat. **3** (1956), 149–157. (Spanish)

Expository paper, in continuation of those in same Rev. **3** 39–44 (1953),; 67–75 (1954); 109–118 (1955) [MR **15**, 421; **17**, 481]. It presents the fundamental properties of elliptic functions of second order. *E. Grosswald.*

González, Mario O. Theory of elliptic functions. Rev. Soc. Cubana Ci. Fís. Mat. **4** (1957), 3–32. (Spanish)

As indicated in an earlier installment [same Rev. **3** (1953), 39–44; MR **15**, 421] the author proposes to develop the general theory of elliptic functions, by taking as fundamental function $w = w(z; \lambda) = \mathrm{Tan}\, z$, obtained by the inversion of the integral $z = \int_0^w \varphi(w; \lambda)dw$, with $\varphi(w; \lambda) = (1 + 2\lambda w^2 + w^4)^{-1}$ and an arbitrary path of integration. In the present installment the author studies the properties of $\mathrm{Tan}\, z$ and related "generalized trigonometric" functions $\mathrm{Sen}\, z$, $\mathrm{Cos}\, z$ and their reciprocals. $\mathrm{Tan}\, z$ contains as particular cases both the trigonometric and the hyperbolic tangents, corresponding to the values $\lambda = 1$ and $\lambda = -1$, respectively. Let $\pm\alpha$, $\pm 1/\alpha$ be the zeros of φ^{-2}; then $\mathrm{Tan}\, z$ has the periods $2K = \int_{C_1} \varphi(w)dw$ and $2K' = -i\int_{C_2} \varphi(w)dw$, with the simply closed curves C_1 and C_2 each including two zeros of φ^{-2}, namely α, $-1/\alpha$ and α, $1/\alpha$, respectively. If λ is real and $y/x = w$, then $\frac{1}{2}\mathrm{Tan}^{-1}\, w$ is the area enclosed between the real axis, an arc γ of $\varphi^{-2} = 0$ in the first quadrant and the radius vector to the point (x, y) on γ, in generalization of the classical cases $\lambda = \pm 1$. $\mathrm{Cos}\, z$ and $\mathrm{Sen}\, z$ are defined by

$$\mathrm{Cos}^4 z + 2\lambda\, \mathrm{Cos}^2 z\, \mathrm{Sen}^2 z + \mathrm{Sen}^4 z = 1, \quad \mathrm{Sen}\, z/\mathrm{Cos}\, z = \mathrm{Tan}\, z.$$

The derivatives and graphs of these functions are studied in detail, for real λ. For complex λ, the paper discusses the inversion of the integral, the periods, zeros, poles, residues at poles, addition formulae and power series expansions of the direct and inverse functions. Sample results:

$$\mathrm{Tan}(u+v) = \frac{\mathrm{Tan}\, u\, \mathrm{Sec}^2 v + \mathrm{Tan}\, v\, \mathrm{Sec}^2 u}{1 - \mathrm{Tan}^2 u\, \mathrm{Tan}^2 v},$$

$$\mathrm{Tan}\, \frac{K}{2} = 1, \quad \mathrm{Tan}[u + (2m+1)K + (2n+1)iK'] = -\mathrm{Tan}\, u.$$

While the present approach could hardly be expected to lead to any new results, its neatness and elegance makes it aesthetically and pedagogically appealing.

E. Grosswald (Philadelphia, Pa.).

Fempl, Stanimir. Sur une combinaison linéaire des intégrales elliptiques normales de première et seconde espèces. Srpska Akad. Nauka. Zb. Rad. **50**, Mat. Inst. 5 (1956), 61–116. (Serbo-Croatian. French summary)

L'objet de ce travail est l'étude de l'expression

$$\Lambda(k, \varphi) = FE(k', \varphi) + EF(k', \varphi) - FF(k', \varphi),$$

où F et E désignent les intégrales elliptiques normales, complètes, de première et seconde espèce de module k, $F(k', \varphi)$ et $E(k', \varphi)$ les mêmes intégrales de module complémentaire $k' = \sqrt{(1-k^2)}$ et d'amplitude φ. L'auteur établit le théorème de multiplication suivant. Toutes les fois que la suite des amplitudes φ_ν satisfait aux conditions

$$\cos \varphi_{\nu+1} = \cos \varphi_1 \cos \varphi_\nu - \sin \varphi_1 \sin \varphi_\nu \sqrt{(1 - k'^2 \sin^2 \varphi_{\nu+1})}$$

on a

$$\Lambda(k, \varphi_m) = m\Lambda(k, \varphi_1) - Fk'^2 \sin \varphi_1 \sum_1^{m-1} \sin \varphi_\nu \sin \varphi_{\nu+1}$$

et inversement. On donne une suite d'applications de cette formule et d'interprétations géométriques (le cône elliptique). *M. Tomić* (Beograd).

Eichler, M. Eine Verallgemeinerung der Abelschen Integrale. Math. Z. **67** (1957), 267–298.

This is a fundamental, definitive paper to which a review can scarcely do justice; yet it is all based on an explicit manipulation due to G. Bol [Abh. Math. Sem. Univ. Hamburg **16** (1949), nos. 3–4, 1–28; MR **11**, 437, 872]. Let $\varphi(\tau)$ be an automorphic form of degree $-n$ under the linear transformation ε, i.e., $\varphi(\tau)[\varepsilon]^{-n}=\varphi(\tau)$, where $\varepsilon\tau=(a_\varepsilon\tau+b_\varepsilon)/(c_\varepsilon\tau+d_\varepsilon)$ and

$$\varphi(\tau)[\varepsilon]^{-n}\equiv\varphi(\varepsilon\tau)(c_\varepsilon\tau+d_\varepsilon)^{-n}(a_\varepsilon d_\varepsilon-b_\varepsilon c_\varepsilon)^{n/2}.$$

Then if $\Phi(\tau)$ is any $(n-1)$-fold integral of $\varphi(\tau)$,

$$\Phi(\tau)[\varepsilon]^{n-2}-\Phi(\tau)\equiv\Omega_\varepsilon(\tau) \tag{1}$$

is an ordinary polynomial of degree $n-2$ in τ. In analogy to abelian integrals, this polynomial is called a "period". Referring to a fuchsian group $\Gamma=\{1, \varepsilon, \eta, \cdots\}$ in the upper half plane, a Riemann surface is defined as usual by identifying points, etc. The group has consistency relations of the type

$$\Omega_{\varepsilon\eta}-\Omega_\varepsilon[\eta]^{n-2}-\Omega_\eta=0. \tag{2}$$

Here $\varphi(\tau)$ is defined conventionally as being of "first or second type" according to the absence or presence of poles. Furthermore, $n\geqq 2$ and is even. Then every automorphic form of the first or second kind leads to a cohomology class in the mapping of Γ into the polynomial module (via Ω), namely the quotient of the cocycles defined by (2) by the coboundaries trivially defined by taking $\Phi(\tau)$ as a polynomial of degree $n-2$ in (1). The interesting thing is the (converse) correspondence of each such cohomology class to a $\varphi(\tau)$ of first or second kind. The author establishes this by using the defining topology of the Riemann surface for an explicit calculation of the rank of said cohomology group. Integrals of the type

$$I(\varphi, \psi)=\int\Phi(\tau)\psi(\tau)d\tau/(2\pi i), \tag{3}$$

where ψ is an arbitrary automorphic form, are used to build up the desired function coordinatewise from the cohomology class.

The author's ultimate purpose is number-theoretic, and he uses integrals of type (3) to calculate the trace on the space of automorphic forms as produced by certain "modular-group" correspondences of the Riemann surface into itself [cf. Eichler, J. Indian Math. Soc. (N.S.) **20** (1956), 163–206; MR **19**, 18].

The author cites another independent source of the underlying formal manipulations in the work of A. Weil [Abh. Math. Sem. Hamburg. Univ. **11** (1935), 110–115]. {The reviewer can cite two additional independent sources rooted in perturbation-theory (where $n=4$). One is M. Schiffer's and D. C. Spencer's work on the variation kernel or quadratic differential [e.g., Functionals of finite Riemann surfaces, Princeton, 1954, p. 316 ff.; MR **16**, 461]. The other is the reviewer's work involving period polynomials whose coefficients are real [Trans Amer. Math. Soc. **82** (1956), 117–127; MR **18**, 194].} *H. Cohn.*

Myrberg, P. J. Über eine Klasse von automorphen Funktionen mehrerer Variablen, die vermittels periodischer Funktionen darstellbar sind. I. Kommutative Gruppen. Ann. Acad. Sci. Fenn. Ser. A. I. no. **235** (1957), 10 pp.

Myrberg, P. J. Über eine Klasse von automorphen Funktionen mehrerer Variablen, die vermittels periodischer Funktionen darstellbar sind. II. Die nichtkommutativen Gruppen. Ann. Acad. Sci. Fenn. Ser. A. I. no. **238** (1957), 16 pp.

The functions under discussion are the automorphic functions of a group Γ of transformations S_ν: $x'=x+\omega_\nu$, $y'=y+\tau_\nu(x)$, ω_ν constants, $\tau_\nu(x)$ entire functions. Introducing a new variable $z=y-\varphi(x)$, where $\varphi(x+\omega_1)-\varphi(x)=\tau_1(x)$, puts Γ into a simpler form. Apart from special cases, if Γ is commutative it can be reduced to a group of transformations $x'=x+\omega_\nu$, $z'=z+\eta_\nu$, η_ν constants, with $\omega_1=2\pi i$, $\omega_2=0$, $\eta_1=0$, $\eta_2=2\pi i$. Then if Γ is generated by N transformations its automorphic functions are determined by $f(x+2\pi i, z)=f(x, z)$ if $N=1$ and by $f(x+2\pi i, z)=f(x, z+2\pi i)=f(x, z)$ if $N=2$. For $N=3$ there are meromorphic, but not entire, automorphic functions. For $N=4$ automorphic functions exist only under certain conditions. When Γ is not commutative the situation is much more complicated and the groups having automorphic functions are fairly restricted.

H. S. Zuckerman (Seattle, Wash.).

Christian, Ulrich. Zur Theorie der Modulfunktionen n-ten Grades. Math. Ann. **133** (1957), 281–297.

A function is here defined to be a modular function if it is invariant and properly behaved (meromorphic) when the fundamental region has been compactified. It is proved that every $n(n+1)/2+1$ modular functions are algebraically dependent over the complex numbers. Every modular function is the quotient of two modular forms and can be expressed rationally in terms of generalized Eisenstein series. *H. S. Zuckerman.*

Khriptun, M. D. On a class of integral functions. Dokl. Akad. Nauk SSSR (N.S.) **113** (1957), 1002–1005. (Russian)

The author announces some properties of the function

$$U_p^{(1)}(z)=\sum_{n=0}^{\infty}\frac{(z/3)^{p+3n}}{(2n)!\Gamma(n+p+1)},$$

which satisfies

$$z^3U'''(z)+\tfrac{3}{2}z^2U''(z)-\tfrac{1}{2}z(6p^2+3p+1)U'(z)+(2p^3+3p^2-\tfrac{1}{4}z^3)U(z)=0$$

and so is a kind of generalized Bessel function. Sets of independent solutions of the differential equation are given. If $p\neq k/3$ or $(2k-3)/6$, they are $U_{p+3/2}^{(2)}(z)$ and $U_{-2p}^{(3)}(z)$, where $U_p^{(2)}$ has denominators $(2n+1)!\Gamma(n+p)$ and $U_p^{(3)}$ has denominators $\Gamma(-p)n!/\Gamma(-2n-p)$. There are generating relations for integral p:

$$e^{zt/3}\cosh(\tfrac{1}{3}zt^{-1})=\sum_{-\infty}^{\infty}t^pU_p^{(1)}(z),$$

$$t^{\frac{1}{2}}e^{zt/3}\sinh(\tfrac{1}{3}zt^{-1})=\sum_{-\infty}^{\infty}t^pU_{p+1}^{(2)}(z),$$

and various integral representations for arbitrary p. There is an addition formula

$$U_0^{(1)}(\lambda(r_1-r_2))=\sum_{m=-\infty}^{\infty}U_m^{(1)}(\lambda r_1)U_{-m}^{(1)}(-\lambda r_2)+\sum_{m=-\infty}^{\infty}U_{m+1}^{(2)}(\lambda r_1)U_{-m+2}^{(2)}(-\lambda r_2),$$

with a generalization to arbitrary subscripts.

R. P. Boas, Jr. (Evanston, Ill.).

Voronin, È. S. On solutions of Weber's equation. Vestnik Moskov. Univ. Ser. Mat. Meh. Astr. Fiz. Him. **12** (1957), no. 1, 8–10. (Russian)

Every solution u of Weber's equation

$$u''+(n+\tfrac{1}{2}-z^2/4)u=0$$

(n positive integer) can be expressed in the form $u=aD_n(z)+bC_n(z)$, where $D_n(0)=0$, $D_n'(0)\neq 0$, $C_n(0)\neq 0$, $C_n'(0)=0$ for n odd resp. $D_n(0)\neq 0$, $D_n'(0)=0$, $C_n(0)=0$, $C_n'(0)\neq 0$ for n even. The function $D_n(z)$ is classical [see Erdélyi, Oberhettinger, and Tricomi, Higher transcendental functions, vol. 2, McGraw-Hill, New York, 1953; MR 15, 419]. The author investigates the properties of the function $C_n(z)$. In particular, asymptotic expansions are given. Finally, there is a table which gives the values of $C_n(z)$ up to 5 places for $z=0(0.2)5$ and $n=0, 1, 2, 3$.

L. Gårding (Lund).

See also: Analytic Theory of Numbers: Hua. **Functions of Complex Variables:** Raljević. **Quantum Mechanics:** Petiau.

Sequences, Series, Summability

Rehtman-Ol'šanskaya, P. G. On a statement of Academician A. A. Markov. Uspehi Mat. Nauk (N.S.) **12** (1957), no. 3(75), 181–187. (Russian)

This paper discusses what might be called an approximate moment problem. Markov's statement, made in 1895, was that a necessary and sufficient condition for the existence of a nonnegative function f for which

$$\alpha_k' < \int_0^l x^k f(x)\,dx < \alpha_k'',$$

$k=0, 1, \cdots, n-1$, is the positivity of a certain sequence of determinants. The author proves a more general statement of the same character.

R. P. Boas, Jr.

Dorleijn, M. Convergent sequences in sequence spaces. Nederl. Akad. Wetensch. Proc. Ser. A. **60**=Indag. Math. **19** (1957), 254–260.

This paper is concerned with some special properties of sequence spaces [for an account of these, and for definitions of dual space, perfect space, normal spaces, and coordinate-cgt., $\alpha\beta$-cgt, $\alpha\beta$-bd, strong $\alpha\beta$-cgt sequences, see the reviewer's "Infinite matrices and sequence spaces" [Macmillan, London, 1950; MR **12**, 694], Ch. 10; this book is referred to below as "I.M."].

The following definitions are introduced in the paper. (α, β) is said to have the property C if every $\alpha\beta$-bd sequence in α which is c-cgt is also $\alpha\beta$-cgt. The set α' associated with α is the set V of sequences u which have the property that to every u of V there exists at least one sequence of positive integers n_i ($n_i<n_{i+1}$, $i=1, 2, \cdots$) such that to every x in α corresponds a number $M=M(u, x)>0$ with

$$(1)\qquad \left|\sum_{j=n_i+1}^{n_{i+1}} u_j x_j\right| < M \quad (i=1, 2, \cdots).$$

The set $\bar{\alpha}$ conjugated with α is the set of all those points u in α' such that, for every sequence n_i for which (1) holds, we also have $\lim_{i\to\infty}\sum_{j=n_i+1}^{n_{i+1}} u_j x_j=0$ for every x in α.

The following results are then proved. (I) The set α' is normal if α is normal. (II) $\alpha^*\leqq\bar{\alpha}\leqq\alpha'$, where α^* is the dual space of α. (III) A necessary and sufficient condition that for every pair (α, β) $\alpha\geqq\phi$, where ϕ is the space of all finite sequences, β normal) the property C holds, is $\beta'=\bar{\beta}$. (IV) $\omega'=\bar{\omega}=\omega^*=\phi$ [the author omits to state what ω is; it is presumably intended to be the space of all sequences, denoted in I.M., p. 273, by σ]. (V) $\phi'=\bar{\phi}=\phi^*=\omega$. (VI) $\sigma_r'=\bar{\sigma}_r=\sigma_\infty$ [For meanings of σ_r, σ_∞, see I.M., p. 273]. (VII) For σ_0, the space of all null sequences [denoted by Z in I.M., p. 273], we have $\sigma_0'=\bar{\sigma}_0=\sigma_\infty$. (VIII) $\sigma_\infty'\neq\bar{\sigma}_\infty$. (IX) For σ, the space of all convergent sequences [denoted by Γ in I.M., p. 273], we have $\sigma'\neq\bar{\sigma}$. (X) For E_r, $r>0$, [I.M. p. 274], we have $E_r'\neq\bar{E}_r$. (XI) For F_r, $r>0$, [I.M., p. 274], we have $F_r'=\bar{F}_r=E_r$.

The paper concludes with a few applications of these results.

R. G. Cooke (London).

Rényi, A. Mathematical notes. II. On the sequence of generalized partial sums of a series. Publ. Math. Debrecen **5** (1957), 129–141.

Als verallgemeinerte Teilsummenfolge $\{A_n\}$ einer gegebenen Reihe $\sum a_n$ wird hier die Folge $A_0=0$, $A_n=a_{k_1}+a_{k_2}+\cdots+a_{k_j}$ bezeichnet, wenn $n=2^{k_1}+2^{k_2}+\cdots+2^{k_j}$ ($k_1>k_2>\cdots>k_j\geqq 0$) die dyadische Darstellung einer natürlichen Zahl n bedeutet. $\{A_n\}$ umfaßt daher in lexikographischer Anordnung alle endlichen Summen von Elementen a_n. (1) Genau dann existiert C_k-lim $A_n=\alpha_1$ ($k=1$ oder 2), wenn $\sum a_n=2\alpha_1$ ist. (2) Genau dann existieren C_1-lim $(A_n)^r=\alpha_r$ für alle $r=1, 2, \cdots$, wenn $\sum a_n$ und $\sum a_n^2$ konvergieren. Für α_r ergibt sich die Darstellung $\alpha_r=\int_0^1 (A(t))^r dt$, wobei $A(t)=\sum a_n\varepsilon_{n+1}(t)$ und $\varepsilon_n(t)=0$ oder 1 die n-te Ziffer in der Dualbruchentwicklung von $t\in\langle 0, 1)$ ist; speziell ist $\alpha_2=\frac{1}{4}[(\sum a_n)^2+\sum a_n^2]$. (3) Bezeichnet $F(x)$ die Verteilungsfunktion von $A(t)$, und ist $N_n(x)$ die Anzahl der A_k mit $k<n$ und $A_k<x$, so gilt $\lim_{n\to\infty} N_n(x)/n=F(x)$ für jede Stetigkeitsstelle x von $F(x)$.

D. Gaier (Stuttgart).

Bosanquet, L. S.; and Tatchell, J. B. A note on summability factors. Mathematika **4** (1957), 25–40.

Sind V_1 und V_2 zwei Summierungsverfahren, so heißen die Zahlen ε_n Faktoren vom Typ (V_1, V_2), wenn für jede V_1-summierbare Reihe $\sum a_n$ die Reihe $\sum a_n\varepsilon_n$ V_2-summierbar ist. Diese Arbeit charakteristiert die Faktoren vom Typ (C_k, A) $(k>-1)$ und $(C_k, |A|)$ $(k\geqq -1)$ wie folgt. (a) Die ε_n sind (C_k, A)-Faktoren $(k\geqq 0)$ genau dann, wenn (i) $\varepsilon_n=\text{Const}+o(1)$ und (ii) $\sum n^k|\Delta^{k+1}\varepsilon_n|<\infty$ gelten; für $-1<k<0$ fällt (i) weg. (b) Die ε_n sind $(C_k, |A|)$-Faktoren $(k\geqq -1)$ genau dann, wenn (iii) $\sum |\varepsilon_n|/n<\infty$ und (ii) gelten. Dies zeigt, daß für $k\geqq 0$ die (C_k, A)- mit den (C_k, C_l)-Faktoren $(l\geqq k)$ und die $(C_k, |A|)$- mit den $(C_k, |C_l|)$-Faktoren $(l\geqq k+1)$ [Peyerimhoff, Acad. Serbe Sci. Publ. Inst. Math. **8** (1955), 139–156; **10** (1956), 1–18; MR **17**, 1076; MR **18**, 651] identisch sind. Verschiedene Teilaussagen von (a) und (b) waren schon bekannt [Bosanquet, Proc. London Math. Soc. (2) **50** (1948), 295–304; MR **10**, 112; Tatchell, J. London Math. Soc. **29** (1954), 207–211; MR **15**, 697]. Daß (iii) notwendig ist für $(C_{-1}, |A|)$-Faktoren, wird mit funktionalanalytischen Mitteln bewiesen [vgl. Peyerimhoff, Math. Z. **55** (1951), 23–54; MR **13**, 933].

D. Gaier (Stuttgart).

See also: Functions of Real Variables: Dvoretzky. **Functions of Complex Variables:** Orts. **Trigonometric Series and Integrals:** Katayama. **Integral Transforms:** Lubich.

Approximations, Orthogonal Functions

Tournarie, Max. Pratique de l'inversion du produit de composition de fonctions histographiées. C. R. Acad. Sci. Paris **245** (1957), 1301–1303.

A continuation of the article [same C.R. **242** (1956), 2509–2512] reviewed in MR **18**, 33.

Korovkin, P. P. On the order of the approximation of functions by linear positive operators. Dokl. Akad. Nauk SSSR (N.S.) **114** (1957), 1158-1161. (Russian)

The author observes that if $L_n(f)$ $(n=1, 2, \cdots)$ are bounded positive linear operators which map the space of continuous functions on $[a, b]$ into the subspace of polynomials of degree not exceeding n, then for at least one of the functions $f(x)=1, x, x^2$ the sequence $n^2\|L_n(f)-f\|$, does not converge to zero. He illustrates by examples the known fact that for positive singular integrals a low degree of approximation usually occurs for all non-linear functions. *G. G. Lorentz* (Detroit, Mich.).

Talalyan, A. A. On the convergence almost everywhere of subsequences of partial sums of general orthogonal series. Akad. Nauk Armyan. SSR. Izv. Fiz.-Mat. Estest. Tehn. Nauki **10** (1957), no. 3, 17–34. (Russian. Armenian summary)

A sequence of functions $\{f_n\}$ defined on [0, 1], measurable (Lebesgue) and finite a.e. is said to be complete in the sense of convergence a.e. if any measurable function on [0, 1] is the pointwise limit a.e. of a sequence of finite linear combinations of elements from $\{f_n\}$.

Theorem: If $\{\varphi_n\}$ is an O.N. set in $L_2[0, 1]$, complete in the sense of convergence a.e., then there exists a series $\sum_1^\infty a_n\varphi_n$ with a_n real and tending toward zero such that for any measurable function $f(x)$ it is possible to find a sequence $\{n_k\}$ for which $\lim_{k\to\infty} S_{n_k}(x)=f(x)$ a.e. on [0, 1], where $S_{n_k}=\sum_{n=1}^{n_k} a_n\varphi_n$.

If the condition $a_n\to 0$ is removed, then it is possible to prove the same theorem for any sequence of functions complete in the sense of convergence a.e. The author also gives necessary and sufficient conditions, in terms of completeness conditions in the topology of $L_2[0, 1]$, that a sequence of functions be complete in the sense of convergence a.e., and also shows it is possible to remove any finite set from a complete sequence and still retain the completeness.

These results generalize the results of Marcinkiewicz [Ann. Soc. Polon. Math. **16** (1938), 84–96]. *A. Devinatz* (St. Louis, Mo.).

Gagaev, B. M. Orthogonal systems of functions whose derivatives are also orthogonal. Uspehi Mat. Nauk (N.S.) **12** (1957), no. 2(74), 133–136. (Russian)

Given weight functions p and q on an interval $[a, b]$, the author extends work of himself [Izv. Akad. Nauk SSSR. Ser. Mat. **10** (1946), 197–206; MR **8**, 263] and D. C. Lewis [Rend. Circ. Mat. Palermo (2) **2** (1953), 159–168; MR **15**, 789] by showing that if completeness is not required of the orthogonal systems it is always possible to construct from any system of functions $\{\varphi_m\}$, $m=1, 2, \cdots$, which are orthogonal with respect to the weight function q and in L^2 of the weight function p a sequence $\{\psi_n\}$ of linear combinations such that the ψ_n are orthogonal with respect to q and the derivatives ψ_n' are orthogonal with respect to p. *M. M. Day* (Urbana, Ill.).

See also: Functions of Real Variables: Timan and Timan. **Functions of Complex Variables:** Geronimus; Kaz'min; Evgrafov and Soloviev.

Trigonometric Series and Integrals

Konyuškov, A. A. On a class of functions. Uspehi Mat. Nauk (N.S.) **12** (1957), no. 4(76), 177–180. (Russian)

Consider the class S_1 [A. Broman, Uppsala thesis, 1947; MR **9**, 182] of real integrable periodic functions whose Fourier coefficients a_n, b_n satisfy $\sum n(a_n^2+b_n^2)<\infty$. It was known that $S_1\subset \mathrm{Lip}(\frac{1}{2}, 2)$ and the author points out that $S_1\subset \mathrm{lip}(\frac{1}{2}, 2)$. The author examines the structure of elements of S_1 more closely. With the obvious norm, S_1 is a Hilbert space. Let $\phi(h)$ be nondecreasing, positive, with $\phi(0+)=0$. Put

$$U(f, h; \phi)=h^{-1}\phi(h)^{-1}\int_0^{2\pi}|f(x+h)-f(x)|dx.$$

Then the subset of S_1 consisting of elements for which $\limsup U(f, h; \phi)=+\infty$ is residual and contains elements of $\mathrm{lip}\,\frac{1}{2}$. The set $L_\infty\cap S_1$ is (in S_1) of first category, and an F_σ but not a G_δ. Let C be the Banach space of continuous periodic functions and give $C\cap S_1$ the norm $\|f\|_C+\|f\|_{S_1}$. The subset of $C\cap S_1$ for which

$$\limsup \omega(h; f)/\phi(h)=+\infty$$

is residual (ω is the modulus of continuity of f). Finally, the author notes that $\mathrm{lip}(\frac{1}{2}, 2)$ is an $F_{\sigma\delta}$ of first category in $\mathrm{Lip}(\frac{1}{2}, 2)$. *R. P. Boas, Jr.* (Evanston, Ill.).

Ul'yanov, P. L. On a class of analytic functions. Uspehi Mat. Nauk (N.S.) **12** (1957), no. 3(75), 229–234. (Russian)

This continues the author's investigations on A-integration and trigonometric series [cf. Mat. Sb. N.S. **35(77)** (1954), 469–490; MR **16**, 467]. Let B denote the class of real sequences which approach zero and are of bounded variation. The author proves the following theorems. (1) If $\{a_n\}\in B$ then the harmonic function $\sum_1^\infty a_k r^k\cos k\phi$ is the A-Poisson integral of the (not necessarily L-integrable) function $\sum_1^\infty a_k\cos k\phi$. (2) If the real and imaginary parts of $\{c_n\}$ belong to B then the function F defined by $\sum c_n z^n$, $|z|<1$, is the A-Cauchy integral of $\sum c_n\zeta^n$, $|\zeta|=1$. (3) If ϕ is analytic in $|z|<1$ and of bounded variation on $|z|=1$, while F satisfies the hypotheses of (2), then $(A)\text{-}\int F(\zeta)\phi(\zeta)d\zeta=0$ $(|\zeta|=1)$. *R. P. Boas, Jr.*

Timan, A. F. Remarks on trigonometric polynomials and Fourier-Stieltjes series. Uspehi Mat. Nauk (N.S.) **12** (1957), no. 2(74), 175–183. (Russian)

Stečkin showed [same J. **10** (1955), no. 1(63), 159–166; MR **16**, 815] that for $K_n(t)=\frac{1}{2}a_0+\sum_{k=1}^n a_k\cos kt$ the constant C in the inequality

$$\int_0^\pi |K_n(t)|dt\geqq C\left|\sum_{k=0}^n \frac{a_k}{n-k+1}\right|$$

does not exceed $2/\pi$, while it is precisely $2/\pi$ for non-negative a_k. The author first remarks that for general (complex) a_k it is true that

$$\int_0^\pi |K_n(t)dt|\geqq \frac{2}{\pi}\left|\sum_{\nu=0}^\infty \frac{1}{2\nu+1}\sum_{k=(2n+1)\nu}^{(2n+1)(\nu+1)-1}\frac{a_{|(2n+1)\nu+n-k|}}{k+\frac{1}{2}}\right|,$$

where $2/\pi$ cannot be replaced by anything larger [equivalent to a theorem of Fejes, J. London Math. Soc. **14** (1939), 44–46; for similar results see van der Corput and Visser, Nederl. Akad. Wetensch., Proc. **49** (1946), 383–392=Indag. Math. **8** (1946), 238–247; MR **8**, 148]. Consequently, if $a_k\geqq 0$ and $a_k\neq 0$,

$$\int_0^\pi |K_n(t)|dt>\frac{2}{\pi}\sum_{k=0}^{2n}\frac{a_{|n-k|}}{k+\frac{1}{2}}.$$

The author then shows that for $m=2[\frac{1}{2}(n+2)]$ or $m=2[\frac{1}{2}(n+1)]+1$,

$$\int_0^\pi |K_n(t)|dt\geqq \frac{2}{\pi}\left|\sum_{\nu=0}^\infty \frac{1}{2\nu+1}\sum_{k=\nu m+1}^{(\nu+1)m-1} k^{-1}a_{|(2\nu+1)m-2k|}\right|$$

and derives some consequences.

Now let $c_k=\frac{1}{2}(a_k-ib_k)$ be the Fourier-Stieltjes coefficients of a function F of bounded variation. The author shows that there is an absolute constant M ($\frac{1}{9}\leqq M<.59$) such that

$$\left|\sum_{k=1}^{n} k^{-1}(a_{n-k}-a_{n+k})\right|\leqq M \operatorname*{var}_{0\leq x\leq 2\pi} F(x),$$

and that if F is absolutely continuous

$$\sum_{k=1}^{n} k^{-1}(a_{n-k}-a_{n+k})=O[\omega_1(F', 1/n)],$$

where ω_1 is the integral modulus of continuity of F. The proof of this theorem also yields a result on multipliers λ_n that preserve for a trigonometric series the property of being a Fourier series, or the Fourier series of a bounded function, or of a continuous function, or a Fourier-Stieltjes series: in each case $\sum k^{-1}(\lambda_{|k-n|}-\lambda_{k+n})$ converges uniformly in n.

Let $\lambda_k^{(n)}$ be a matrix each row of which is a set of Fourier-Stieltjes coefficients. Let f be continuous with Fourier coefficients a_n, b_n, and form

$$f_n(x)\sim\tfrac{1}{2}a_0\lambda_0^{(n)}+\sum_{k=1}^{\infty}\lambda_k^{(n)}(a_k\cos kx+b_k\sin kx).$$

Then if $f_n(x)\to f(x)$ uniformly for each continuous f, it follows that $\lambda_k^{(n)}\to 1$ as $n\to\infty$; $\lambda_k^{(n)}=O(1)$; and

$$\sum k^{-1}[\lambda_{|k-\nu|}^{(n)}-\lambda_{k+\nu}^{(n)}]$$

converges uniformly in ν with its sum uniformly bounded in n. *R. P. Boas, Jr.* (Evanston, Ill.).

Katayama, Miyoko. Fourier Series. XIII. Transformation of Fourier series. Proc. Japan Acad. 33 (1957), 75–78.

Soit $\Lambda=(\lambda_{\nu n})$ ($\nu, n=0, 1, \cdots$) une matrice infinie à éléments réels,

$$(*)\qquad a_0/2+\sum(a_\nu\cos\nu x+b_\nu\sin\nu x)=\sum A_\nu(x)$$

a série de Fourier de $f(x)$. Cette serie (*) sera Λ-sommable vers $\alpha(x)$ si $\sum\lambda_{\nu n}A_\nu(x)$ converge pour tout n vers $\alpha_n(x)$ et si $\lim_{n\to\infty}\alpha_n(x)=\alpha(x)$ existe. Alors, si $f(x)\in L^p$ ($p>1$), les conditions: (1) $\lim\lambda_{\nu n}$ existe pour tout n; (2) $K_n(t)=\lambda_{0n}/2+\sum\lambda_{\nu n}\cos\nu t\in L^q$ ($1/p+1/q=1$); (3) $\int_{-\pi}^{\pi}|Kn(t)|dt=O(1)$ pour tout n, sont des conditions nécessaires et suffisantes pour Λ-sommabilité de (*). Si $f(x)\in L$, (2) doit être remplacé par $|K_{\mu n}(t)|\leqq M_n$ (où $K_{\mu n}(t)$ désigne la somme partielle de K_n) et (3) par $|K_n(t)|\leqq M$ pour tout n et t. On donne ensuite les conditions suffisantes pour Λ-sommabilité de (*) portant sur les coefficients $\lambda_{\nu n}$ directement. *M. Tomić* (Beograd).

Yano, Kenji. On Hardy and Littlewood's theorem. Proc. Japan Acad. 33 (1957), 73–74.

Si f est continue et si $\int_0^{2\pi}|f(x+h)-f(x)|^{1/\alpha}dx=O(h)$ quand $h\to 0$ ($0<\alpha\leqq 1$), la série de Fourier de f est sommable $(C, -\alpha+\delta)$ pour tout $\delta>0$. *J. P. Kahane.*

See also: Integral Transforms: Lubich; Ogievetzki. **Elasticity, Plasticity:** Ignaczak.

Integral Transforms

Lubich, Yu. I. Some Tauberian theorems for generalized Fourier transforms. Dokl. Akad. SSSR (N.S.) **113** (1957), 32–35. (Russian)

Consider functions α which are of bounded variation on every finite interval and satisfy

$$\limsup_{|a|\to\infty}|a|^{-n}\operatorname*{Var}_{a\leq\mu\leq a+1}\alpha(\mu)<\infty.$$

Such a function has a generalized Fourier-Stieltjes transform à la Bochner, of order $[n+2]$, and the author wishes to infer information about α from information about the kth derivative of the transform. His results are extensions of V. A. Marčenko's [Izv. Akad. Nauk SSSR. Ser. Mat. **19** (1955), 381–422; MR **17**, 852] to the case of nonintegral n and k. *R. P. Boas, Jr.* (Evanston, Ill.).

Pollard, Harry; and Standish, Charles. Inversion of a class of discrete convolution transforms. Scripta Math. **22** (1956), 207–216 (1957).

For $1>a_1\geqq a_2\geqq\cdots 0$; $1>b_1\geqq b_2\geqq\cdots 0$ and $\sum_1^\infty(a_j+b_j)<\infty$, let

$$(1)\qquad \sum_{n=-\infty}^{\infty}k(n)z^n=\frac{\prod_{j=1}^{\infty}(1-a_j)(1-b_j)}{\prod_{j=1}^{\infty}(1-a_jz)(1-b_j/z)}.$$

Consider the discrete convolution

$$(2)\qquad f(n)=\sum_{-\infty}^{\infty}k(n-m)g(m)$$

with $k(\cdot)$ as kernel. The authors show that if for a given $g(m)$ (2) converges (conditionally) for some n, then it converges (conditionally) for all n, and that (2) may be inverted by the formula

$$\lim_{N\to\infty}P_N(\delta)f(n)=g(n);\quad P_N(\delta)=\frac{\prod_{j=1}^{N}(1-a_j\delta)(1-b_j/\delta)}{\prod_{j=1}^{\infty}(1-a_j)(1-b_j)}.$$

Here $\delta f(n)=f(n-1)$, $(1/\delta)f(n)=f(n+1)$. The kernels (1) are a subclass of the set of totally positive sequences [see I. J. Schoenberg, Courant anniversary volume, Interscience, New York, 1948; MR **9**, 337]. These considerations also afford a simple and interesting introduction to the material of the "The convolution transform" [Princeton, 1955; MR **17**, 478] by the reviewer and D. V. Widder. *I. I. Hirschman, Jr.* (St. Louis, Mo.).

Srivastava, Krishna Ji. Fractional integration and Meijer transform. Math. Z. **67** (1957), 404–412.

The author studies the Meijer transform

$$F(s)=\int_0^\infty e^{-st/2}W_{k+\frac{1}{2},m}(st)(st)^{-k-\frac{1}{2}}f(t)dt.$$

A large number of relations (too complicated to reproduce here) are proved which essentially assert that a fractional integral of F is a combination of Meijer transforms of suitable fractional integrals of f. *I. I. Hirschman, Jr.*

Ogievetzki, I. I. Generalisations of some results of G. H. Hardy, J. E. Littlewood and A. Zygmund on fractional integration and differentiation of periodic functions. Ukrain. Mat. Ž. **9** (1957), 205–210. (Russian. English summary)

[In the summary Littlewood's initials are misprinted S.E.] The author establishes theorems about fractional integrals and derivatives and the Lipschitz classes to which they belong by using results from the theory of approximation ("constructive theory of functions"). These include results by Hardy and Littlewood [Math. Z. **27** (1928), 565–606] and Zygmund [Duke Math. J. **12** (1945), 47–76; MR **7**, 60] as well as some new ones. [A

similar investigation was undertaken by Králik, Acta Math. Acad. Sci. Hungar. 7 (1956), 49–64; MR 18, 54.] The main results are as follows, where f_α is the αth integral and f^α the αth derivative of the periodic function f, and E_n denotes best approximation by trigonometric polynomials of degree n. (1) $E_n(f)=O(n^{-\alpha})$, $\alpha>0$, implies that f^γ exists and is continuous for $\gamma<\alpha$, $(f^\gamma)_\gamma=f$, and $E_n(f^\gamma)=O(n^{-\alpha+\gamma})$. (2) $E_n(f)=O(n^{-\alpha})$ implies $E_n(f_\gamma)=O(n^{-\alpha-\gamma})$. (3) $f^\beta\in\Lambda^*$ is equivalent to $E_n(f)=O(n^{-\gamma-\beta})$. (4) $f^\beta\in\mathrm{Lip}\,\alpha$ implies $(f_\gamma)^\delta\in\mathrm{Lip}\,(\alpha+\beta+\gamma-\delta)$ if $\alpha+\beta+\gamma-\delta<1$; $(f_\gamma)^\delta\in\Lambda^*$ if $\alpha+\beta+\gamma-\delta=1$; $(f^\gamma)_\delta\in\mathrm{Lip}\,(\alpha+\beta+\delta-\gamma)$ if $0<\alpha+\beta+\delta-\gamma<1$ and $\gamma<\alpha+\beta$; $(f^\gamma)_\delta\in\Lambda^*$ if $\alpha+\beta+\delta-\gamma=1$, $\gamma<\alpha+\beta$; similar results for $f^\beta\in\Lambda^*$, $f_\beta\in\mathrm{Lip}\,\alpha$, $f_\beta\in\Lambda^*$. An essential tool is an inequality of Civin's [Duke Math. J. 8 (1941), 656–665; MR 3, 108]. *R. P. Boas, Jr.*

Lauwerier, H. A. A note on the formulae of Plemelj. Math. Centrum Amsterdam. Afd. Toegepaste Wisk. Rep. **TW 39** (1957), i+10 pp.

If we assume that

$$\phi^+(z)=\frac{1}{2\pi i}\int_{-\infty}^{\infty}\frac{f(t)}{t-z}\,dt \quad (\mathrm{Im}\,z>0),$$

$$\phi^-(z)=\frac{1}{2\pi i}\int_{-\infty}^{\infty}\frac{f(t)}{t-z}\,dt \quad (\mathrm{Im}\,z<0),$$

and that

$$\lim_{y\downarrow 0}\phi^+(x+iy)=\phi^+(x),\ \lim_{y\uparrow 0}\phi^-(x+iy)=\phi^-(x),$$

then under certain circumstances

$$\phi^+(x)-\phi^-(x)=f(x),$$

$$\phi^+(x)+\phi^-(x)=\frac{1}{\pi i}\int_{-\infty}^{\infty}\frac{f(t)}{t-x}\,dt.$$

The author discusses various sufficient conditions for the above formulae to hold. Unfortunately, he does little more than prove theorems 95, 98, 91, 92, 93 and 105 of Titchmarsh, "Introduction to the theory of Fourier integrals" [Oxford, 1937].

Reading is made difficult firstly, by the use of $f_{-\infty}^{\infty}$ to represent both an ordinary L-integral and a mean square integral, and secondly, by the fact that the norm in the main part of the paper is not in the form defined in § 1. *J. L. Griffith* (Kensington).

See also: Analytic Theory of Numbers: Mikolás. **Functions of Complex Variables:** Falgas. **Topological Vector Spaces:** Ishihara. **Banach Spaces, Banach Algebras, Hilbert Spaces:** Zaidman. **Statistics:** Fox. **Elasticity, Plasticity:** Muki. **Classical Thermodynamics, Heat Transfer:** Hofmann.

Ordinary Differential Equations

★ **Kneschke, A. Differentialgleichungen und Randwertprobleme. Lehrbuch für Naturwissenschaftler und Ingenieure. Bd. I. Gewöhnliche Differentialgleichungen.** VEB Verlag Technik, Berlin, 1957. 504 pp.

This book contains many methods of practical importance. There are no existence proofs and the discussion is confined to the real field. There is considerable emphasis on approximation, e.g., by the Runge-Kutta method. Expansions in series of eigenfunctions are also treated in detail.

★ **Эл'гольц, Л. Э. [Èl'sgol'c, L. È.] Дифференциальные уравнения. [Differential equations.]** Gosudarstv. Izdat. Tehn.- Teor. Lit., Moscow, 1957. 272 pp. 5.70 rubles.

This textbook is intended for students in mathematics and physics in the state universities in the USSR. There are six chapters on the topics: differential equations of first order, of higher order, systems of equations, theory of stability, equations with deviating argument, partial equations of first order. There are exercises with solutions.

★ **Spiegel, Murray R. Applied differential equations.** Prentice-Hall Mathematics Series. Prentice-Hall, Inc., Englewood Cliffs, N. J., 1958. xv+381 pp. $6.75.

The purpose of the book is "to provide in an elementary manner a reasonable understanding of differential equations for students of engineering, physics, and chemistry, as well as students of mathematics who are interested in applications of their field." There are ten chapters, the first eight on ordinary and the last two on partial differential equations.

★ **Rainville, Earl D. A short course in differential equations.** 2nd ed. The Macmillan Company, New York, 1958. x+259 pp. $4.50.

The first edition was published in 1949. The book is intended for a first course in differential equations. The omission of infinite series permits greater discussion of applications. Partial differential equations are not included.

★ **Rainville, Earl D. Elementary differential equations.** 2nd ed. The Macmillan Company, New York. 1958. xii+449 pp. $5.50.

A much fuller treatment than that in the book reviewed above, with an extensive discussion of methods based on power series and Fourier series.

★ **Erdélyi, A. Asymptotic solutions of differential equations with transition points.** Proceedings of the International Congress of Mathematicians, 1954, Amsterdam, vol. III, pp. 92–101. Erven P. Noordhoff N.V., Groningen; North-Holland Publishing Co., Amsterdam, 1956. $7.00.

This is an expository report on the asymptotic theory of differential equations of the form $d^2y/dx^2-\nu^2f(x,\nu)y=0$ for large values of the parameter ν. The emphasis is on asymptotic representations valid in a region containing a transition point, i.e., a point where $f(x,\infty)$ vanishes. The basic ideas of the techniques developed by R. E. Langer and by T. M. Cherry are presented in some detail. *W. Wasow* (Madison, Wis.).

Tartakovskiĭ, V. A. The spectrum of the matrix of differentiation at a point of stationarity. Vestnik Leningrad. Univ. Ser. Mat. Meh. Astr. **12** (1957), no. 7, 52–67. (Russian. English summary)

Let $\lambda_1, \lambda_2, \cdots, \lambda_n$ be the characteristic numbers of the linear approximation of a canonical autonomic system of n differential equations of first order with a point of stationarity in the origin O of the system of the space coordinates $x_1, \cdots, x_n$. Then all the characteristic numbers of the matrix of differentiation of that system of equations are representable in the form

$$(*)\qquad \lambda_{m_1m_2\cdots m_n}=m_1\lambda_1+m_2\lambda_2+\cdots+m_n\lambda_n,$$

where $m_1, m_2, \cdots, m_n$ are arbitrary non-negative integer

numbers not all simultaneously equal to zero. The system of all such numbers (*) forms the spectrum of the matrix of differentiation.

The characteristic polygon of the point of stationarity 0 of our system of equations is the minimal convex polygon which contains all the numbers $\lambda_1, \lambda_2, \cdots, \lambda_n$. A point of stationarity is named primitive when the spectrum of the matrix of differentiation at that point has no limit points. It is proved (§ 4) that it is necessary and sufficient for the primitivity of a point of stationarity that its characteristic polygon does not contain that point of stationarity itself. § 3 contains a definition of a system of elementary divisors of a matrix of differentiation at a primitive point of stationarity. §§ 2, 5 contain some theorems about the elementary divisors of "scale-matrices", i.e., block-matrices whose blocks, situated above the block-diagonal, are all equal to zero. *Author's summary.*

Wintner, Aurel. Energy dissipation and linear stability. Quart. Appl. Math. **15** (1957), 263–268.

The author proves that every solution of $x''=F(t)x$ with $F(t)$ of class C^1 on $[t_0, \infty)$ satisfies $x(t)=O(1)$, $x'(t)=O(1)$ as $t\to\infty$ provided (1) F is symmetric and its maximum eigenvalue $\mu(t)$ satisfies $\mu(t)\leqq-\varepsilon<0$, and either (2) $F'(t)$ is non-negative definite or (3) the minimum eigenvalue $\alpha(t)$ of $F'(t)$ is such that $-\int^\infty \min[0, \alpha(t)]dt<\infty$. If in addition, $\mu(t)\to-\infty$ as $t\to+\infty$, then (1) and (2) imply $x(t)=o(1)$ as $t\to\infty$. The proofs depend upon considerations of the energy function $(x', x')-(x, F(t)x)$ along a solution $x(t)$. These results generalize earlier results of the author [Amer. J. Math. **68** (1946), 553–559; MR 8,272] and Ważewski [Studia Math. **10** (1948), 48–59; MR **10**, 40] and others for systems $x'=A(t)x$.

{It appears to the reviewer that (2) and (3) should be stated as above; in the paper it is supposed that in (2) $F'(t)$ be non-positive and in (3) the maximum eigenvalue $\beta(t)$ of $F'(t)$ satisfy $\int^\infty \max[0, \beta(t)]dt<\infty$.}

H. A. Antosiewicz (Providence, R.I.).

Conti, Roberto. Sistemi differenziali asintoticamente equivalenti. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) **22** (1957), 588–592.

The equations (1) $\dot{y}=A(t)y$ with $A(t)$ measurable and summable over every finite interval of $[T, +\infty)$, and (2) $\dot{x}=A(t)x+g(t, x)$ with $g(t, x)$ lipschitzian in x for $\lambda(t)\geqq0$, where $\int_T^\infty \lambda(t)dt<\infty$ and $\int_T^\infty \|g(t, 0)\|dt<\infty$, are said to be asymptotically equivalent if there is a homeomorphism ω of $t=T$ into itself such that if $\eta=\omega(\xi)$ then $\|x(t, \xi)-y(t, \eta)\|\to0$ as $t\to+\infty$. The author proves that (*) uniform asymptotic stability of $y(t)\equiv0$ implies asymptotic equivalence of (1) and (2). He previously proved the same conclusion under hypotheses which ensure the uniform asymptotic stability of $y(t)\equiv0$ [cf. Ann. Mat. Pura Appl. (4) **41** (1956), 95–104; MR **18**, 482]. When (2) is linear, i.e., (2') $\dot{x}=B(t)x\equiv A(t)x+[B(t)-A(t)]x$, it follows that (*) holds if $\int_T^\infty \|B(t)-A(t)\|dt<\infty$. More generally, if $y(t)\equiv0$ is uniform-asymptotically stable, then (1) is asymptotically, equivalent to every (2') to which it is t_∞-similar [cf., e.g., Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) **19** (1955), 247–250; MR **18**, 483] by a matrix $T(t)$ for which $\lim_{t\to\infty} T(t)=T(\infty)$ exists and is non-singular.

H. A. Antosiewicz (Providence, R.I.).

Repin, Yu. M. On stability of solutions of equations with retarded argument. Prikl. Mat. Meh. **21** (1957), 253–261. (Russian)

The author considers the system

$$\dot{x}_i=X_i[t, x_1(t-\tau_1(t)), \cdots, x_1(t-\tau_m(t)), \cdots, x_n(t-\tau_1(t)), \cdots, x_n(t-\tau_m(t))] \quad (1\leqq i\leqq n),$$

where $X_i(t, \xi_1, \cdots, \xi_{mn})$ is defined, continuous and lipschitzian in ξ for $t\geqq A$, $\|\xi\|<H$, $X_i(t, 0)\equiv0$, and $\tau_j(t)$ is defined, continuous, non-negative and uniformly bounded for $t\geqq A$. He defines for the trivial solution uniform asymptotic stability and total stability ("stability under constantly acting perturbations"), and proves that the former implies (1) the latter, and (2) that every solution starting sufficiently close to the origin decreases exponentially in norm as $t\to\infty$. His proofs are direct, without reference to consideration of Lyapunov functions. *H. A. Antosiewicz* (Providence, R.I.).

Putnam, C. R. On future and past stability in incompressible systems. J. Math. Mech. **6** (1957), 669–672.

In the conservative system (*) $dx/dt=f(x)$, where x and f are vectors, let $f(x)$ be of class C^1 and satisfy div $f\equiv0$ on an invariant open set. A solution $x=x(t)$ of (*) is called (Liapounoff) stable for $t\geqq0$ (or for $-\infty<t<\infty$) if, for every $\varepsilon>0$, there exists a $\delta=\delta_\varepsilon>0$ such that if $x=y(t)$ is any solution of (*) satisfying $|y(0)-x(0)|<\delta$, then $y(t)$ is defined and satisfies $|y(t)-x(t)|<\varepsilon$ for $t\geqq0$ (or for $-\infty<t<\infty$). It is shown that if $x=x(t)$ is a Bohr almost periodic solution of (*) which is stable for $t\geqq0$, then it is stable for $-\infty<t<\infty$. *P. Hartman.*

Ziemba, Stefan. Free vibration with damping of marked non-linear character. Arch. Mech. Stos. **9** (1957), 525–548. (Polish and Russian summaries)

After a brief exposition of qualitative methods for the study of the differential equation

$$Ay''+[B+F(y')]y'+Cy=0,$$

the author reviews results on the zeros of the solutions [G. Sansone, Equazioni differenziali nel campo reale, t. 2] 2ª ed., Zanichelli, Bologna, 1949, pp. 356–374; MR **11**, 32, and then shows that the results are true for soft as well as hard damping. *J. P. LaSalle* (Notre Dame, Ind.).

Reissig, Rolf. Über eine nichtlineare Differentialgleichung zweiter Ordnung. Wiss. Z. Humboldt-Univ. Berlin. Math.-Nat. Reihe **6** (1956/57), 1–2.

In the system of real differential equations

$$\dot{u}=v, \quad \dot{v}=E(t)-G(u)-F(v)$$

it is assumed that $G(u)$ and $F(v)$ are strictly monotone increasing and $E(t)$ is periodic. Further conditions on E, F and G are assumed to guarantee the existence and uniqueness of solutions of the system. It is observed that the usual fixed point proof for the existence of a periodic solution is valid when

$$\lim_{v\to\infty}(F(v)-F(-v))>\operatorname{Max} E(t)-\operatorname{Min} E(t)$$

and when $G(u)$ takes on a certain pair of values dependent upon $E(t)$ and $F(v)$. *W. R. Utz* (Columbia, Mo.).

Colombo, Giuseppe. Sulle oscillazioni non-lineari di combinazione. Rend. Sem. Mat. Univ. Padova **27** (1957), 162–175.

The author considers the equation

$$x''+x-E\sin\omega t=\varepsilon f(t, x, x', \varepsilon), \tag{1}$$

where $\omega\neq1$ (either rational or irrational), $f(t+2\pi/\omega, x, x', \varepsilon)=f(t, x, x', \varepsilon)$, f is continuous together with its first derivatives with respect to x, x', $|f(t, x, x', \varepsilon)|<M(a, b)$ (in-

dependent of t, ε) if $|x|<a$, $|x'|<b$, $|\varepsilon|<\varepsilon_0$. Assuming a solution of (1) of the form

$$x=A\cos(t+\theta)+\frac{E}{1-\omega^2}\sin\omega t+ \tag{2}$$
$$\varepsilon\varphi(t, A, \theta)+\varepsilon^2\psi(t, A, \theta, \varepsilon),$$

the author generalizes the standard perturbation procedure for finding periodic solutions of (1), and gives conditions for the determination of $A=A(\varepsilon, \theta)$ so that (2) is a solution of (1) for ε sufficiently small. Conditions are also given for the stability of this solution. *J. K. Hale.*

Halanay, Aristide. Solutions périodiques des systèmes non-linéaires à petit paramètre. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) **22** (1957), 30–32.

Let $\dot{x}=X_0(x, t)+\mu X_1(x, t)$, X_0, X_1 continuous and periodic in t with period ω, X_0 analytic in x; let $X_0^{(m)}(x, t)$, $m\geqq 2$, be the algebraic form consisting of the terms of lowest degree ($=m$) in the power series development of X_0. Then, if the vector field $\omega^{-1}\int_0^\omega X_0^{(m)}(x, t)dt$ has an isolated singular point at $x=0$ whose index is $\neq 0$, for sufficiently small $|\mu|$ there is at least one periodic solution with period ω in the neighborhood of the origin. *J. L. Massera.*

Berstein, I. On the problem of periodic solutions of nonlinear systems with a small parameter. Dokl. Akad. Nauk SSSR (N.S.) **113** (1957), 9–11. (Russian)

Take the n-vector system

$$\dot{x}=X(x, t)+\mu Y(x, t, \mu),$$

where (a) X and Y are continuous with period ω in t and defined in $|x|<\rho$; (b) in any cylinder $|x|\leq\sigma$, $0\leq t\leq T$, the solutions are unique and continuous in the initial conditions and in μ; (c) we have $X(0, t)\equiv 0$ under the above circumstances, and $x=0$ is an asymptotically stable solution of $\dot{x}=X(x, t)$. Then for $|\mu|\leq\mu_0$ the system possesses a periodic solution disposed in the neighborhood of $x=0$. *S. Lefschetz* (Mexico City).

Landis, E. M.; and Petrovskiĭ, I. G. On the number of limit cycles of the equation $dy/dx=P(x, y)/Q(x, y)$, where P and Q are polynomials. Mat. Sb. N.S. **43(85)** (1957), 149–168. (Russian)

The highly interesting results announced in Dokl. Akad. Nauk SSSR (N.S.) **113** (1957), 748–751, [MR **19**, 35] are fully proved in one present paper. The method is closely related to that of the authors' paper for $N=2$ [Mat. Sb. **37(79)** (1955), 209–250; MR **17**, 364]. *S. Lefschetz.*

Molchanov, N. N. The use of the theory of continuous groups of transformations in investigating the solutions of ordinary differential equations. Dokl. Akad. Nauk SSSR (N.S.) **112** (1957), 998–1001. (Russian)

Let X and Y be rational polynomials in x and y, and consider the equation $dy/dx=Y(x, y)/X(x, y)$. This note gives an upper bound for the number of limit cycles. *M. M. Day* (Urbana, Ill.).

Nemyckij, V. V. Some problems in the qualitative theory of differential equations. (Survey of contemporary literature.) Časopis Pěst. Mat. **81** (1956), 451–469. (Czech)

A translation from the Russian of the article in Uspehi Mat. Nauk (N.S.) **9**, no. 3(61), (1954), 39–56 [MR **16**, 130].

Sargsyan, I. S. Asymptotic behavior of the derivatives of a spectral function of a Sturm-Liouville operator. Akad. Nauk Armyan. SSR. Izv. Fiz.-Mat. Estest. Tehn. Nauki **10** (1957), no. 3, 3–16. (Russian. Armenian summary)

The author considers the equation $y''-q(x)y=\lambda y$ on the half-line $(0, \infty)$, where q is real and summable on each finite subinterval. The differential equation, together with an initial condition at the origin, is assumed to give a problem with non-negative spectrum. An asymptotic formula is given for the Riesz means of the kth derivative of the spectral function of this problem, under the hypothesis that q has a $(k-1)$st derivative which is summable on each finite subinterval. This extends some previous work of the author [Dokl. Akad. Nauk SSSR (N.S.) **104** (1955), 821–824; MR **17**, 853]. *E. A. Coddington.*

Agranovich, Z. S.; and Marchenko, V. A. Re-establishment of the potential from the scattering matrix for a system of differential equations. Dokl. Akad. Nauk SSSR (N.S.) **113** (1957), 951–954. (Russian)

The authors consider the equation (A) $y''+\lambda^2 y=V(x)y$ for n-dimensional vectors y on $0<x<\infty$. The potential matrix V is assumed to be hermitian and to satisfy (*) $\int_0^\infty t|V(t)|dt<\infty$, where $|V(t)|=\max_j\sum_{k=1}^n|V_{jk}(t)|$. A number of results are stated concerning the determination of the potential matrix V from the scattering matrix. The equation (1) $Y''+\lambda^2 Y=V(x)Y$, for n by n matrices Y on $0<x<\infty$, has for $\operatorname{Im}\lambda\leq 0$ a solution $E(x, \lambda)$ representable in the form

$$E(x, \lambda)=e^{-i\lambda x}I+\int_x^\infty K(x, t)e^{-i\lambda t}dt,$$

where I is the identity matrix, and the matrix $K(x, t)$ satisfies $|K(x, t)|\leq C\sigma(\frac{1}{2}(x+t))$; C a constant, and $\sigma(x)=\int_x^\infty|V(t)|dt$. Also $2K(x, x)=\int_x^\infty V(t)dt$, $0<x<\infty$. The boundary problem defined by (A) and the condition (B) $y(0, \lambda)=0$ is considered. Let $G(x, \lambda)$ be the matrix solution of (1) satisfying $G(0, \lambda)=0$, $G'(0, \lambda)=I$. The problem (A), (B) has a finite number p of eigenvalues of the form λ_k^2, $\lambda_k=-i\mu_k$, $\mu_k>0$, and the normalized eigenvectors are the columns of the matrix $U(x, \lambda_k)=E(x, \lambda_k)M_k$. Here M_k is a hermitian matrix whose rank is equal to the multiplicity of the eigenvalue λ_k^2; it is called a normalization matrix. For real $\lambda\neq 0$ one has $U(x, \lambda)=E(x, -\lambda)-E(x, \lambda)S(-\lambda)=2iG(x, \lambda)[E^*(0, -\lambda)]^{-1}$, where $S(\lambda)$ is a unitary matrix, called the scattering matrix. It possesses the following properties: (I_S) the matrix $I-S(\lambda)$ is the Fourier transform of a hermitian matrix $F_1(u)$, where

$$F_1(u)=\frac{1}{2\pi}\int_{-\infty}^\infty[I-S(\lambda)]e^{i\lambda u}d\lambda;$$

the elements of F_1 are summable on $(0, \infty)$, and on $(-\infty, 0)$ are representable as the sum of two functions, one of which is summable, and the other bounded and square summable; (II_S) the equation

$$-x(t)+\int_{-\infty}^0 x(\xi)F_1(t+\xi)d\xi=0 \qquad (-\infty<t\leq 0),$$

has no non-zero solution; (III_S) for any $u>0$, $F'(u)$ exists, and $\int_0^\infty u|F'(u)|du<\infty$. The normalization matrices M_k and the eigenvalues λ_k^2 are such that: (IV) the equation

$$x(t)+\int_0^\infty x(\xi)F(t+\xi)d\xi=0 \qquad (0\leq t<\infty),$$

where

$$F(u)=\sum_{k=1}^p M_k^2 e^{-\mu_k u}+F_1(u),$$

has no non-zero solution; (V) the number of linearly independent solutions of

$$x(t)+\int_0^\infty x(\xi)F_1(t+\xi)d\xi=0 \quad (0\leq t<\infty),$$

is equal to the sum of the ranks of the matrices $M_1, \cdots, M_p$.

The basic result stated is that the conditions I_S, II_S, III_S, IV, and V are necessary and sufficient in order that a unitary matrix $S(\lambda)$, hermitian matrices M_k, and numbers $-\mu_k^2$, $\mu_k>0$, be respectively the scattering matrix, normalization matrices, and eigenvalues of some boundary problem (A), (B) with a potential matrix V satisfying (*). The authors further state that if $S(\lambda)$ is a unitary matrix satisfying I_S and II_S, then it is possible in an infinite number of ways to choose numbers $\mu_k>0$ and hermitian matrices M_k so that all the conditions I_S, II_S, IV, and V are valid. In particular, the properties I_S, II_S, III_S are necessary and sufficient in order that a unitary matrix $S(\lambda)$ be the scattering matrix of a boundary problem (A), (B) with a potential matrix satisfying (*).

E. A. Coddington (Princeton, N.J.).

Cameron, R. H. Differential equations involving a parametric function. Proc. Amer. Math. Soc. **8** (1957), 834–840.

The author determines conditions on the function $f(t, u)$, under which the differential system $dz/dt+f[t, y(t)+z(t)]=0$, $z(0)=0$, $t\in I$, has a solution $z(t)$ on the unit interval I for almost all choices of the function y in the space C, where C denotes the space of functions which are continuous on I and which vanish at $t=0$, and where "almost all" means all except a set of Wiener measure zero. (As the author mentions, the differential system goes into the equivalent nonlinear integral equation $y(t)=x(t)+\int_0^t f[s, x(s)]ds$, $t\in I$, under the transformation $z(t)=x(t)-y(t)$.) A major point of interest in the author's theorem is the phrase "almost all". He gives a relatively simple set of conditions on f which do not force the integral equation to have a solution $x\in C$ for every $y\in C$, but which do ensure that it have such a solution for almost every choice of y in C. He gives a counterexample to show that his theorem can apply in cases where the classical theorems (i.e., theorems giving conditions under which there are solutions for all choices of y in C) do not apply. The proof of the theorem makes use of a result by R. H. Cameron and J. M. Shapiro [Ann. of Math. (2) **62** (1955), 472–497; MR **17**, 632].

W. T. Martin.

Krasnoselsky, M. A.; Krein, S. G.; and Sobolevsky, P. E. On differential equations with unbounded operators in Hilbert space. Dokl. Akad. Nauk SSSR (N.S.) **112** (1957), 990–993. (Russian)

The authors continue their study [same Dokl. (N.S.) **111** (1956), 19–22; MR **19**, 550] of the differential equation $dx/dt+A(t)x=f(t)$, where $f(t)$ is a given function from the closed real interval $[0, b]$ to Hilbert space H, and $A(t)$ is a given function from $[0, b]$ to the set of non-negative linear operators in H.

M. M. Day (Urbana, Ill.).

Mikusiński, J. Équations différentielles à coefficients constants considérées dans les espaces linéaires généraux. Bull. Acad. Polon. Sci. Cl. III. **4** (1956), 137–139.

This note announces without proof some theorems about the number of linearly independent solutions of equations of the type described in the title. Details are contained in the paper reviewed below. There is no topology assumed, and differentiation is a postulated operation. An equation of order n can have fewer than n independent solutions. The main theorem deals with conditions which insure the existence of n solutions.

A. E. Taylor (Los Angeles, Calif.).

Mikusiński, J. Sur les solutions linéairement indépendantes des équations différentielles à coefficients constants. Studia Math. **16** (1957), 41–47.

Etude abstraite des équations différentielles à coefficients constant. Soit E un espace vectoriel sur un corps C de caractéristique 0. Soit D un opérateur linéaire sur E, appelé dérivation, tel que, pour tout polynome P de degré n à une indéterminée à coefficients dans C, l'équation différentielle $P(D)x=0$, $x\in E$, ait au plus n solutions indépendantes. Soit d'autre part Λ un opérateur linéaire sur E, tel que $D(\Lambda x)=\Lambda Dx+x$. Par exemple, C sera le corps des complexes, E l'espace vectoriel des fonctions analytiques d'une variable réelle t, D la dérivation usuelle, Λ la mulplication par t.

Soit alors P un polynome, $\prod_i P_i^{k_i}$ sa décomposition en produit de polynomes irréductibles. (1) L'équation différentielle $P_i(D)x=0$ a exactement $s_i=0$ ou n_i solutions indépendantes, où n_i est le degré de P_i. Si $s_i=n_i$, soient $x_{i,p}$ de telles solutions indépendantes. 2) L'équation différentielle $P_i^{k_i}(D)x=0$ a exactement k_is_i solutions indépendantes; si $s_i=n_i$, on pourra prendre les $\Lambda^j x_{i,p}$ $(0\leq j\leq k_i-1)$. 3) L'équation différentielle $P(D)x=0$ a exactement $\sum_i k_is_i$ solutions indépendantes, qui peuvent s'obtenir en prenant les précédentes pour les diverses valeurs de i.

Ces résultats s'appliquent en particulier si C est le corps des opérateurs de Mikusiński [voir, e.g., Studia Math. **12** (1951), 227–270; MR **13**, 751], E l'espace des fonctions indéfiniment dérivables d'une variable réelle t à valeurs dans C, D la dérivation en t, et Λ la multiplication par t.

L. Schwartz (Paris).

Krein, S. G. On correctness classes for certain boundary problems. Dokl. Akad. Nauk SSSR (N.S.) **114** (1957), 1162–1165. (Russian)

Let $A(t)$ be an unbounded operator in a Hilbert space defined on the interval $I=[0, T]$. The problem consisting of the equation (*) $dx/dt+A(t)x=0$ and the initial condition $x(0)=x_0$ is correct on I in the function class $\mathfrak{M}$ if for every $\varepsilon>0$ and $t\in I$ there is a $\delta(\varepsilon, t)$ for which the inequality $\|x(0)\|<\delta$ implies $\|x(t)\|<\varepsilon$ for every solution $x(t)$ of (*) in $\mathfrak{M}$. As the tool for testing against this definition the author proves the following estimate for the norm:

$$\|x(t)\|\leq\|x(0)\|^{1-\alpha(t)}\|x(T)\|^{\alpha(t)},$$

where $\alpha(t)=(\exp kt-1)/(\exp kT-1)$, under the hypotheses that $A(t)$ is selfadjoint, its domain of definition $D(A)$ does not depend on t, and the strong derivative dA/dt exists for which $(dA/dt\,x, x)\leq k(Ax, x)$, $x\in D$, $t\in(0, T)$. It follows that (i) if $A(t)$ is normal for $t\in I$ and if $\mathrm{Re}(dA/dt\,x, x)\leq k\,\mathrm{Re}(Ax, x)$, then the above problem is correct on I in the class $\mathfrak{M}$ of functions uniformly bounded on I; (ii) if A is a constant normal operator whose spectrum does not include zero, then the problem $d^2x/dt^2-Ax=0$, $x(0)=x_0$, $x'(0)=x_0'$ is correct in the class $\mathfrak{M}$ of functions with uniformly bounded derivative on I. These results are used to affirm the correctness in the class L_2 of certain boundary-value problems involving equations of parabolic and elliptic type.

R. N. Goss (San Diego, Calif.).

See also: Linear Algebra: Browne. **Functions of Real Variables:** Cogan and Norman. **Special Functions:** Voronin. **Topological Vector Spaces:** Bogdanov. **Manifolds, Connections:** Haimovici. **Numerical Methods:** Fettis. **Control Systems:** Letov.

Partial Differential Equations

Lions, J. L. Contribution à un problème de M. M. Picone. Ann. Mat. Pura Appl. (4) **41** (1956), 201–219.

Lors du Second Colloque sur les équations aux dérivées partielles tenu à Bruxelles en 1954 [Thone, Liège, 1955], M. Picone a posé la question de l'existence et de l'unicité de la solution d'un problème d'élasticité rencontré dans la construction des barrages [ibid. pp. 9–11]. Il s'agit en fait de rechercher l'état d'équilibre d'un milieu élastique borné noyé dans un autre milieu de constantes élastiques différentes (le barrage et le rocher). Dans le présente article, l'auteur pose le problème de M. Picone sous une forme généralisée (il se place dans R^n et prend les conditions aux limites en moyenne) et le résout en appliquant les méthodes générales d'un travail antérieur [Acta Math. **94** (1955), 13–153; MR **17**, 745], ce qui en montre, une fois de plus, la fécondité. *H. G. Garnir* (Liège).

Stampacchia, Guido. Osservazioni sull'esistenza e sull'unicità della soluzione dei problemi al contorno misti per equazioni a derivate parziali del secondo ordine di tipo ellittico. Rend. Accad. Sci. Fis. Mat. Napoli (4) **22** (1955), 144–148 (1956).

L'A. dimostra un teorema di unicità per il problema misto relativo all'equazione (non lineare)

$$E(u)=g(x, u) \quad (x \in T),$$

essendo E l'operatore ellittico auto aggiunto del secondo ordine, ed essendo anegnate le condizioni (non lineari) di frontiera:

$$\{u(M)\}_{F_1T}=\alpha(M), \quad \left\{\frac{\partial u(M)}{\partial \nu_M}\right\}_{F_2T}=\phi(M, u(M)),$$

$$(FT=F_1T+F_2T).$$

Alle condizioni già enunciate dall' A. per l'esistenza [Ann. Mat. Pura Appl. (4) **40** (1955), 193–209; MR **19**, 750] l'A. aggiunge, per ottenere l'unicità, le condizioni $g_u(x, u) \geqq 0$, $\phi_u(x, u) \geqq 0$. *L. Amerio* (Milano).

Capriz, Gianfranco; e Lesky, Peter. Su due problemi singolari di Dirichlet. Consiglio Naz. Ricerche. Pubbl. Ist. Appl. Calcolo no. **371**, 7 pp. (1953).

Szmydt, Z. Sur le problème Goursat concernant les équations différentielles hyperboliques du second ordre. Bull. Acad. Polon. Sci. Cl. III. **5** (1957), 571–575, XLIX. (Russian summary)

Let $f(x, y, z, p, q)$ be continuous for $|x|$, $|y| \leq \alpha$ and arbitrary z, p, q and satisfy a uniform Lipschitz condition with respect to (z, p, q). Let $\gamma(t)$, $\delta(t)$, $\sigma(t)$, $\tau(t)$ be of class C^1 for $|t| \leq \alpha$; $\sigma(0)=\tau(0)$; $|\gamma(t)|$, $|\delta(t)| \leq \alpha$; $\gamma(0)=\delta(0)=0$; and $(\max |\gamma'(t)|)(\max |\delta'(t)|)<1$. The author gives a proof for the following: if $\beta>0$ is sufficiently small, the problem $z_{xy}=f(x, y, z, z_x, z_y)$, $z(x, \gamma(x))=\sigma(x)$, $z(\delta(y), y)=\tau(y)$ has a unique solution $z(x, y)$ for $|x|$, $|y| \leq \beta$, possessing continuous partial derivatives z_x, z_y, z_{xy}. As in the case of the Darboux problem, the existence (but not the uniqueness) statement remains valid if the Lipschitz condition with respect to z is omitted [cf. Hartman and Wintner, Amer. J. Math. **74** (1952), 834–864; MR **14**, 475; Ciliberto, Ricerche Mat. **4** (1955), 15–29; MR **17**, 621; Szmydt, Bull. Acad. Polon. Sci. Cl. III. **4** (1956), 579–584; MR **18**, 741]. The proof depends on an application of the Schauder fixed-point theorem and can be generalized to the case where z and f are vectors. *P. Hartman.*

Szmydt, Z. Sur un problème concernant un système d'équations différentielles hyperboliques d'ordre arbitraire à deux variables indépendantes. Bull. Acad. Polon. Sci. Cl. III. **5** (1957), 577–582, XLIX. (Russian summary)

Let u be a vector and $u_{ij}=\partial^{i+j}u/\partial x^i \partial y^j$. By the methods used in the paper reviewed above, the author obtains existence and uniqueness theorems for solutions of the partial differential equation $u_{mp}=f(x, y, u, \cdots, u_{ij}, \cdots)$, where $0 \leq i+j<m+p$, $0 \leq i \leq m$, $0 \leq j \leq p$, subject to conditions of the form: $u_{mj}(x, y)=g_j(x, u(x, y), \cdots, u_{ik}(x, y), \cdots)$ on an arc $y=\gamma_j(x)$, where $1 \leq j \leq p$, $i+k<m+p$ and $0 \leq i<m$; and $u_{ip}(x, y)=h_i(y, u(x, y), \cdots, u_{kj}(x, y), \cdots)$ on an arc $x=\delta_i(y)$, where $1 \leq i \leq m$, $k+j<m+p$ and $0 \leq j<p$. In these conditions, $y=\gamma_j(x)$ and $x=\delta_j(y)$ are given arcs and $g_j(x, u, u_{ik}, \cdots)$, $h_i(y, u, \cdots, u_{kj}, \cdots)$ are given functions of the independent variables $(x, u, \cdots, u_{ik}, \cdots)$, $(y, u, \cdots, u_{kj}, \cdots)$, respectively.

P. Hartman (Los Angeles, Calif.).

Karapetyan, K. I. On Cauchy's problem for an equation of hyperbolic type degenerating on the initial plane. Dokl. Akad. Nauk SSSR (N.S.) **106** (1956), 963–966. (Russian)

On considère dans $R_x^n \times R_t$, $t>0$, l'opérateur

$$D=D_t^2-\sum \mu_{ij}(x, t)D_{x_ix_j}+\sum a_i(x, t)D_{x_i}+a_0(x, t)D_t+b(x, t),$$

où la forme $Q(p, x, t)=\sum \mu_{ij}(x, t)p_ip_j$ est définie positive pour $t>0$; soit $\tilde{p}(t, x)$ la plus petite valeur caractéristique de cette forme; on peut avoir $\tilde{p}(0, x)=0$ pour certains x. On étudie le problème de Cauchy pour D. L'A. montre que sous certaines hypothèses de régularité sur les coefficients et sur $\tilde{p}(t, x)$, le problème de Cauchy admet une solution unique. Le principe de la démonstration consiste à résoudre le problème de Cauchy pour $t \geqq \varepsilon>0$ et à établir des majorations du type "intégrales d'énergie" qui permettent de passer à la limite.

Ceci généralise Berezin, Mat. Sb. N.S. **24(66)** (1949), 301–320 [MR **11**, 112; **13**, 559]; cf. aussi l'article analysé ci-dessous. *J. L. Lions* (Nancy).

Krasnov, M. L. The mixed boundary problem and the Cauchy problem for degenerate hyperbolic equations. Dokl. Akad. Nauk SSSR (N.S.) **107** (1956), 789–792. (Russian)

The author considers the hyperbolic equation

$$(1) \quad Lu \equiv \frac{\partial^2 u}{\partial t^2}-\sum_{i,k=1}^{m}\frac{\partial}{\partial x_i}\left(a_{ik}(x, t)\frac{\partial u}{\partial x_k}\right)+\sum_{i=1}^{m} b_i(x, t)\frac{\partial u}{\partial x_i}+e(x, t)\frac{\partial u}{\partial t}+d(x, t)u=h(x, t)$$

which is degenerate for $t=0$, where $x=(x_1, \cdots, x_m)$, $a_{ik}=a_{ki}$ and $\sum_{i,k=1}^{m} a_{ik}\xi_i\xi_k \geqq c^2 t^\alpha \sum_{i=1}^{m}\xi_i^2$ (c^2=constant). Let $Q=D \times (0 \leq t<l)$, where D is a bounded region in E^m with boundary Γ. In the problem considered, (1) is to be solved subject to the conditions (2) $u=0$ on $\Gamma \times (0 \leq t<l)$, $u=u_0(x)$ and $\partial u/\partial t=u_1(x)$ for $t=0$. The following results are announced. If $0<\alpha<2$, then, under certain growth

and regularity conditions on the coefficients of (1), the mixed boundary problem (1), (2) has a unique generalized solution in the function space obtained by completing the functions which vanish near Γ and the base of Q and have piecewise continuous first partial derivatives in Q with respect to the metric

$$\int_Q \left(\frac{\partial u}{\partial t}\frac{\partial v}{\partial t} + \frac{1}{t^2\eta}\sum_{i=1}^{m}\frac{\partial u}{\partial x_i}\frac{\partial v}{\partial x_i}\right) dQ,$$

where the constant η, $0<\eta<\frac{1}{2}$ is determined by one of the conditions on the coefficients of (1). The same is true for $\alpha>2$ under another set of conditions on the coefficients. The proof, which is sketched, is based on an analog of Galerkin's method due to M. I. Višik [same Dokl. (N.S.) **97** (1954), 193–196; **99** (1954), 189–192; **100** (1955), 409–412; MR **17**, 859.] Moreover, conditions on the coefficients are given under which the generalized solution of (1), (2) will have L_2 derivatives of various orders in any subregion of Q which is bounded away from Γ and the base of Q. Finally, the author notes that similar results hold for the Cauchy problem for (1), and, by a suitable modification of the method, for both the mixed boundary and Cauchy problems for the Euler-Poisson-Darboux equation. *D. G. Aronson* (Minneapolis, Minn.).

Prodi, Giovanni. Soluzioni periodiche di equazioni a derivate parziali di tipo iperbolico non lineari. Ann. Mat. Pura Appl. (4) **42** (1956), 25–49.

Il presente lavoro è dedicato alla ricerca delle soluzioni periodiche rispetto a t dell'equazione

$$u_{tt}-\Delta_2 u+g(x, t, u_t)=f(x, t, u)$$

in $n+1$ variabili indipendenti. Un'ipotesi fondamentale è quella del carattere "dissipativo" del termine g; essa permette di escludere la presenza di soluzioni proprie (come potremo ancora chiamare le soluzioni relative al caso di f identicamente nulla) e permette di ottenere maggiorazioni a priori per le soluzioni. *Dall'introduzione.*

Nash, John. Parabolic equations. Proc. Nat. Acad. Sci. U.S.A. **43** (1957), 754–758.

This note outlines the proofs of theorems on the continuity of solutions of linear, parabolic and elliptic differential equations. The equations considered are of the forms

$$(1)\qquad \sum_{ij}\frac{\partial}{\partial x_i}\left(C_{ij}\frac{\partial T}{\partial x_j}\right)=\frac{\partial T}{\partial t},$$

$$(2)\qquad \sum_{ij}\frac{\partial}{\partial x_i}\left(C_{ij}\frac{\partial T}{\partial x_j}\right)=0,$$

where C_{ij} is a symmetric real matrix (varying in space and time for (1) or in space for (2)) constrained only by universal upper and lower bounds $c_2\geq c_1>0$ on its eigenvalues.

Th. 1. If $T(x, t)$ is a solution of (1) satisfying $|T|\leq B$ when $t=t_0$, then for $t_2\geq t_1\geq 0$,

$$(3)\qquad |T(x_1, t_1)-T(x_2, t_2)|\leq BA\left[\left(\frac{|x_1-x_2|}{\sqrt{(t_1-t_0)}}\right)^{\alpha}+\left(\frac{t_2-t_1}{t_1-t_0}\right)^{\alpha/(2\alpha+2)}\right],$$

where α and A are constants that depend only on c_1, c_2 and the space dimension n.

Th. 2. If $T(x, t)$ is a solution of (2) in a region and $|T|\leq B$ on the boundary, then

$$|T(x)-(T\bar{x})|\leq BD\left[\frac{|x-\bar{x}|}{\min(d(x), d(\bar{x}))}\right]^{\alpha/(\alpha+1)},$$

where α is the α of (3), $d(x)$ is the distance of x from the boundary of the region and D depends only on c_1, c_2 and n. *E. T. Copson* (St. Andrews).

Lidskiĭ, V. B. A theorem on the spectra of a perturbed differential operator. Dokl. Akad. Nauk SSSR (N.S.) **112** (1957), 994–997. (Russian)

Let $\mathcal{E}_n$ be real Euclidean n-space, let L_2 be the space of all measurable complex functions of summable square on $\mathcal{E}_n$, let L be a differential operator, $Lu=-\Delta u+p_1(x)u$, defined on a subspace $D(L)$ of L_2, where Δ is Laplace's operator in $\mathcal{E}_n$. Assume that $p_1(x)$ is a continuous complex-valued function, that L is closed and that L has the resolvent function $R_\lambda=(L-\lambda E)^{-1}$ defined on the resolvent set $\sigma\subset\mathcal{E}_n$; assume that if $\lambda\in\sigma$, then R_λ is one-to-one from L_2 into $D(L)$. If $p_2(x)$ is continuous on $\mathcal{E}_n$ and tends to zero as x tends to infinity, define a perturbed operator $\tilde{L}$ by $\tilde{L}u=Lu+p_2(x)u$. Th. 1. The resolvent $\tilde{R}_\lambda$ of $\tilde{L}$ is defined in a subset $\bar{\sigma}$ of σ so large that $\sigma-\bar{\sigma}$ is at most countable and has no limit points in σ. The points of $\sigma-\bar{\sigma}$ are poles of the resolvent $\tilde{R}_\lambda$. *M. M. Day.*

Zaikin, D. A.; and Kazarnovskiĭ, M. V. On an approximative method for solution of boundary problems with complicated boundary conditions. Uspehi Mat. Nauk (N.S.) **12** (1957), no. 4(76), 175–176. (Russian)

A variational method is proposed, which differs from that of Ritz. Consider, e.g., a membrane with the unit circle for fixed edge, so that the corresponding eigenvalue problem is:

$$(1)\qquad \Delta u+\lambda u=0.$$

We now seek an nth approximation $u_k^{(n)}$ to the kth eigenfunction u_k in the form

$$(2)\qquad u_k^{(n)}=\sum_{i=1}^{n}A_{ki}^{(n)}\phi_{ki}^{(n)},$$

where $\phi_{ki}^{(n)}$ is the solution of (1) with appropriate boundary conditions chosen for each value of n, k, i. E.g., we may suppose $\phi_{ki}^{(n)}$ vanishes on the boundary of a rectangle, so that (2) can be written

$$(3)\qquad u_k^{(n)}=\sum_{i=1}^{n}A_{ki}^{(n)}\cos(\alpha_{ki}^{(n)}x+\gamma_{ki}^{(n)})\cos(\beta_{ki}^{(n)}y+\delta_{ki}^{(n)}).$$

Then the parameters A, α β, γ, δ for each k are determined from a minimum problem of the form

$$I_k^{(n)}=\min\int_C |u_k^{(n)}|^2 ds$$

with conditions of orthonormality. The method can be generalized immediately to other boundary problems, but questions of convergence present difficulty. *S. H. Gould* (Providence, R.I.).

Mihlin, S. G. On a variational method for the solution of extremal problems. Leningrad. Gos. Univ. Uč. Zap. **144**. Ser. Mat. Nauk **23** (1952), 151–164. (Russian)

The present paper contains a simple variational formulation of boundary value problems of the Dirichlet type, in terms of positive linear operators A defined on a linear subset M of a Hilbert space H (that is, $(Au, u)>0$, $u\in M$, $u\neq 0$). In particular the author treats, as examples of the general theory, the Dirichlet problem for the elliptic equation

$$-\frac{\partial}{\partial x_\alpha}\left(A_{\alpha\beta}\frac{\partial u}{\partial x_\beta}\right)+Bu=f(x_1, \cdots, x_n);\ A_{ik}=A_{ki},$$

subject to the boundary condition $u=0$ on the boundary; and the determination of the displacements in elasticity, when the displacements themselves are prescribed on the boundary. These considerations were developed later in more detail by the author in his book "The problem of the minimum of a quadratic functional" [Gostehizdat, Moscow-Leningrad, 1952; MR **16**, 41]. *J. B. Diaz.*

Glantz, Herbert, and Reissner, Eric. On finite sum equations for boundary value problems of partial difference equations. J. Math. Phys. **34** (1956) 286–297.

The authors attack certain problems involving solution of Laplace's equation in half-spaces with mixed boundary conditions. Ordinarily, one transforms to a singular integral equation; approximate solution of the latter is complicated by the fact that this solution is usually singular. The authors propose to replace the differential problem first by a difference problem. They then derive a finite sum equation analogous to the integral equation. The sum equation yields a set of linear algebraic equations; a solution of this is an approximate solution of the original differential problem. The authors apply the method to a two-dimensional problem whose exact solution is known and to a three-dimensional problem whose exact solution is not known. *M. A. Hyman.*

See also: Ordinary Differential Equations: El'sgol'c; Krein. **Calculus of Variations:** Stampacchia. **Topological Vector Spaces:** Trèves.

Difference Equations, Functional Equations

See: Functions of Real Variables: Cogan and Norman.

Integral and Integrodifferential Equations

See: Fields, Rings: Barnett and Abian. **Partial Differential Equations:** Glantz and Reissner. **Fluid Mechanics, Acoustics:** Eckhaus.

Calculus of Variations

★ **Lusternik, L. A. Kürzeste Linien. Eine Einführung in die Variationsrechnung.** VEB Deutscher Verlag der Wissenschaften, Berlin, 1957. 108 pp. DM 7.60.

A translation by Gerhard Tesch of the Russian book reviewed in MR **17**, 861.

Stampacchia, Guido. Problemi al contorno misti per equazioni del calcolo delle variazioni. Ann. Mat. Pura Appl. (4) **40** (1955), 193–209.

Sia $F(x, u, p)$ con $x\equiv(x_1, \cdots, x_n)$, $p\equiv(p_1, \cdots, p_n)$ una funzione definita e continua, insieme con le derivate parziali dei primi due ordini, per x variabile in un campo limitato T e per tutti i valori finiti di u, $p_1, \cdots, p_n$, e si supponga che la forma quadratica $\sum_{i,j=1}^{n} F_{p_ip_j}(x, u, p)\lambda_i\lambda_j$ sia definita positiva. Siano F_1T e F_2T due porzioni della frontiera di T, le quali esauriscono tale frontiera. Sia $\Phi(x, u)$ una funzione definita su F_2T, la quale è continua in u e misurabile in x rispetto alla funzione di insieme: area su F_2T. Sia A_α la classe delle funzioni $f(x)\equiv f(x_1, \cdots, x_n)$ definite in T, per le quali f è assolutamente continua rispetto a ciascuna delle variabili separatamente, e le derivate parziali $p_i(x)\equiv\partial f/\partial x_i$ $(i=1, \cdots, n)$ sono integrabili (secondo Lebesgue) insieme con $|p_i(x)|^\alpha$, ove $\alpha\geqq 1$; e sia $\bar{A}_\alpha$ la sottoclasse costituita da quelle funzioni di A_α, per le quali $F(x, u(x), p(x))$, (ove $p(x)\equiv(\partial u/\partial x_1, \cdots, \partial u/\partial x_n)$ è integrabile in T, mentre $\Phi(x, u(x))$ è integrabile su F_2T. Si considera il funzionale

$$I(u)=\int_T F(x, u(x), p(x))dx+\int_{F_2T}\Phi(x, u(x))d\sigma \qquad (dx\equiv dx_1\cdots dx_n),$$

supponendo che sia (1) $F(x, u, p)\geqq\mu_1\sum_{i=1}^{n}|p_i|^\alpha+\mu_2|u|^\alpha+N$, (2) $\Phi(x, u)\leqq M_0|u|^\alpha+N(x)$, (3) $\nu(x)|u|^\alpha+M(x)\leqq\Phi(x, u)$, ove è $\mu_1>0$, $\mu_2\geqq 0$, $M_0>0$, mentre $\nu(x)\geqq 0$, $M(x)$, $N(x)$ sono integrabili su F_2T. Mediante la semicontinuità l'A. stabilisce l'esistenza del minimo assoluto di $I(u)$ nell'insieme H delle funzioni della classe $\bar{A}^\alpha$, le quali assumono gli stessi valori prefissati su F_1T, se è verificata una delle seguenti condizioni: a) F_1T non è vuoto; b) F_1T è vuoto, ma ha luogo la (3) con $\nu(x)\geqq\nu>0$ almeno in una porzione di F_2T avente misura positiva; c) F_1T è vuoto; nella (1) è $\mu_2>0$, mentre la (3) può anche non essere verificata. — Per giungere a un secondo teorema di esistenza l'A. rileva che esiste il minimo assoluto del funzionale quadratico $D(u)=\int_T(\sum_{i=1}^{n}p_i^2)dx+\int_\gamma u^2d\sigma$, (ove γ è una porzione regolare di F_2T avente misura positiva) nell'insieme delle funzioni di A_2, che si annullano su F_1T e per le quali è $\int_T u^2dx=1$. Ciò premesso alla (1) viene sostituita la disuguaglianza $F(x, u, p)\geqq\mu\sum_{i=1}^{n}p_i^2-qu^2+f(x)u+P(x)$, ove è $\mu>0$, mentre $f(x)$, $P(x)$ sono funzioni integrabili in T assieme a $f^2(x)$; e le (2) e (3) hanno luogo per $\alpha=2$. Allora esiste il minimo assoluto di $I(u)$, se, oltre alle condizioni indicate nel primo teorema, è rispettivamente: in a): $q<\lambda u$; in b): $q<\lambda\min[\mu, \nu]$; in c): $q<0$; ove λ è il minimo di $D(u)$ e tale minimo è supposto positivo. Altrettanto interessante è la seconda parte del lavoro (per la quale rinviamo al lavoro stesso), ove vengono studiate le proprietà differenziali delle estremanti, rilevando che queste soddisfano quasi dappertutto all'equazione differenziale di Eulero; particolarmente delicato è lo studio del comportamento dell'estremante su F_2T. Infine l'A. perviene a due teoremi di esistenza della soluzione di problemi al contorno misti per equazioni a derivate parziali (non lineari) del secondo ordine di tipo ellittico.

S. Cinquini (Zbl **65** (1956), 337).

Greco, Donato. Su un teorema di calcolo delle variazioni. Ricerche Mat. **5** (1956), 159–166.

The author proves the existence of a minimum for the positive quasi-regular integral

$$\iint_D f(x, y, z, z_x, z_y)dxdy$$

with prescribed boundary values, under essentially the following hypothesis on f:

$$|f(x, y, z, 0, 0|\leqq M, \qquad M>0,$$

$$f(x, y, z, p, q)\geqq\mu(p^2+q^2)+N, \qquad \mu>0,\ N>0.$$

This generalizes results of Cecconi [see the paper reviewep below], Morrey [Univ. California Publ. Math. (N.S.) **1** (1943), 1–130; MR **6**, 180], and other mathematicians. The proof is based on a refinement of a levelling process first used by Lebesgue [Rend. Circ. Mat. Palermo **24** (1907), 371–402] and also by Tonelli [Ann. Scuola Norm. Sup. Pisa (2) **2** (1933), 89–130]. *W. H. Fleming.*

Cecconi, Jaurès. Un teorema sull'esistenza del minimo degli integrali doppi del calcolo delle variazioni, in forma non parametrica. Riv. Mat. Univ. Parma **6** (1955), 45–64.

Let D be an open region in a bounded portion of the (x, y) plane. Let D^* be the frontier of this region and let $z(x, y)$ be a function defined on $D+D^*$, taking on the

continuous boundary values $\phi(x, y)$ on D^*. The author is concerned with minimizing the integral $\mathcal{J}(z, D)=\iint F(x, y, z, p, q)dxdy$ over the class of z's which have the prescribed boundary values and are absolutely continuous in the sense of Tonelli on D^*, under the condition that $m(p^2+q^2)+H<F(x, y, z, p, q)$ for some real H, $m>0$, $p^2+q^2>L$ (given, fixed) >0. F is supposed to have continuous first partial derivatives on D. Problems of this type, i.e., those with $\alpha=2$ in the condition $F(x, y, z, p, q)>(p^2+q^2)^{\alpha/2}$, have been studied previously by Morrey, in sections 5 and 6 of Univ. California Publ. Math. (N.S.) **1** (1943), 1–130, [MR **6**, 180] and by Sigalov [Trudy Moskov. Mat. Obšč. **2** (1953), 201–233; MR **15**, 442].

The author proves an existence theorem, as follows: Suppose that the diameters of the components of D^* are bounded from zero. Suppose that A is a convex bounded set in R^3 containing the graph of the function $\phi(x, y)$, $(x, y) \in D^*$, and containing the graph of at least one ACT function $z=f(x, y)$, $(x, y) \in D+D^*$, taking on the prescribed boundary values and for which the integral is finite. Then there is an ACT function $f_0(x, y)$ whose graph lies in A and which takes on the prescribed (by ϕ) boundary values, which minimizes the integral among all such functions.

This theorem is an improvement of the theorem of Sigalov "in the bounded domain" [see the review of his paper cited above], in that Sigalov requires F to satisfy the "approximation theorem". Otherwise the theorems are essentially identical. However, the author does not discuss the problem in the unbounded domain, which is treated by Sigalov. In the latter case there is no set A.

J. M. Danskin (Princeton, N.J.).

Cecconi, Jaurès. Sulla esistenza del minimo degli integrali del calcolo delle variazioni estesi ad una superficie di forma parametrica. Riv. Mat. Univ. Parma **6** (1955), 163–191.

The author proves a theorem concerning two-dimensional integrals of the calculus of variations in parametric form in 3-space extended over surfaces of the type of the 2-sphere. The usual conditions for surfaces of the type of the 2-cell is that they bound a given Jordan curve; the author's condition for surfaces of the type of the two-sphere is that they "wrap around" a given bounded open set Ω.

The open set Ω is required to satisfy a smoothness condition, satisfied, as the author states, by any set in E_3 whose frontier is a simple surface with a sufficiently regular representation; for instance spheres, ellipsoids, and polyhedra. This condition is as follows: (a) If $\Omega(\delta)$ is the set of points of Ω whose distance from its frontier Ω^* is greater than $\delta>0$, then $\Omega(\delta)$ is connected if $\delta<1/\lambda$; (b) any simple closed polygon p of diameter less than $1/\lambda$ is the boundary of a polyhedron P of the type of the 2-cell, whose area does not exceed $\lambda\times(\text{length } p)^2$ and whose diameter does not exceed $\lambda\times(\text{diam } p)^2$.

An oriented surface Σ is said to wrap around Ω if, for any $x \in \Omega$, the topological index of x with respect to Σ is not zero.

The admissible surfaces are those which are of the type of the 2-cell, have their support in a fixed convex compact A, wrap around Ω, are oriented, and have finite Lebesgue area. $\Omega+\Omega^*$ is supposed interior to A. The integral

$$\mathcal{J}(\Sigma)=(\Sigma)\int F(x, p)$$

is defined as a Weierstrass-Cesari integral. The definition is similar to that given by Cesari for surfaces of the type of the 2-cell.

The principal theorem is as follows: If $\mathcal{J}(\Sigma)$ is quasi-regular (i.e., the Weierstrassian is non-negative), positive definite (i.e., the integrand is positive when $p\neq 0$), and if there is an admissible surface, then there is an admissible surface minimizing $\mathcal{J}(\Sigma)$ among all admissible surfaces.

The proof makes substantial use of a smoothing process due to Cesari [Amer. J. Math. **74** (1952), 265–295; MR **14**, 292].

J. M. Danskin (Princeton, N.J.).

Al'ber, S. I. On the periodic problem of global variational calculus. Uspehi Mat. Nauk (N.S.) **12** (1957), no. 4(76), 57–124. (Russian)

In 1953 the author [Dokl. Akad. Nauk SSSR (N.S.) **91** (1953), 1237–1240; MR **15**, 457], using Lusternik's cohomological method, obtained a new lower bound for the number of geometrically distinct closed geodesics on a thrice differentiable riemannian manifold homeomorphic with the n-sphere. Writing $n=2^k+s$ $(s<2^k)$, his lower bound is $2n-1-s$. For $n=2^k$ this coincides with an estimate of Lusternik [Trudy Mat. Inst. Steklov. **19** (1947); MR **9**, 596]. The present paper gives a detailed proof; more than that, it contains a detailed exposition and amplification of Lusternik's methods. The author defines singularly-spectral homology and cohomology groups for metric spaces by taking limits in the sense of Čogošvili [Soobšč. Akad. Nauk Gruzin. SSR **14** (1953), 583–588; MR **16**, 389] of the corresponding singular groups, using singular simplices of diameter $\leqq\varepsilon_k$, where ε_k describes a sequence tending to 0. Let $C(n)$ denote the space of all circles on the n-sphere. The ring of intersections for $C(n)$ is obtained by a method of Chern [Ann. of Math. (2) **49** (1948), 362–372; MR **9**, 456]. Using this he exhibits $2n-1-s$ suitable singularly-spectral homology classes of $C(n)$ to each of which corresponds a closed geodesic; and by the "principle of subordination of classes" he shows that whenever two of these geodesics have the same length c, then there is a set of closed geodesics of positive dimension with length c. The proof given for this subordination principle leans on Pontrjagin's removing theorem in the general form proved by P. S. Alexandroff [Izv. Akad. Nauk SSSR. Ser. Mat. **6** (1942), 227–282; MR **4**, 249; **5**, 48].

W. H. Fleming (Providence, R.I.).

See also: Partial Differential Equations: Stampacchia.

TOPOLOGICAL ALGEBRAIC STRUCTURES

Topological Groups

Jacoby, Robb. Some theorems on the structure of locally compact local groups. Ann. of Math. (2) **66** (1957), 36–69.

This paper is devoted to proving for locally compact local groups the theorems already known on the structure of locally compact groups. The author makes no pretense of doing anything but adapt the usual proofs for groups to the local case; the difficulties lie mostly in defining precisely the hypothesis of smallness necessary to carry out the old proofs and then fitting the whole together into a coherent proof of the main theorem, viz.: a finite-dimensional metrizable locally compact local group is

locally isomorphic to a global group. From this follows the affirmative solution to the local form of Hilbert's fifth problem: A locally euclidean local group is a local Lie group. Although this has been "known" for some time, the actual proof with full attention to the difficult details is a welcome addition to the literature.

The paper is written throughout in a compact symbolic style. Proofs are omitted whenever they can easily be supplied by one familiar with the corresponding theorem for groups. The paper is, therefore, a good deal longer than its twenty-four pages suggest. *A. M. Gleason.*

Mackey, George W. Les ensembles boréliens et les extensions des groupes. J. Math. Pures Appl. (9) **36** (1957), 171–178.

A set S endowed with a sigma-field of subsets is called a Borel space and the members of the sigma-field are called Borel sets. A map of one Borel space into another is called a Borel map if the inverse image of each Borel set is again Borel. The usual Borel sets form a Borel structure for a topological space, and it is known that the Borel structures so induced on any uncountable Borel subset of a complete separable metric space are isomorphic. Such a Borel structure, or the unique separating Borel structure on a denumerable space is called standard.

A group G endowed with a Borel structure is called a Borel group if the group mapping $(x, y)\to xy^{-1}$ of $G\times G\to G$ is a Borel map. If a left-invariant sigma-finite measure is defined on the Borel sets of a standard Borel group, then it is isomorphic as a Borel group to a uniquely determined, metrizable, locally compact group. This result is applied to the problem of group extension for topological groups. If a factor system for a locally compact group Q with values in a locally compact abelian group K is a Borel function from $Q\times Q$ to K, then the corresponding group extension has a unique locally compact topology satisfying the desirable conditions. Proofs are sketched.

A. M. Gleason (Cambridge, Mass.).

Mackey, George W. Borel structure in groups and their duals. Trans. Amer. Math. Soc. **85** (1957), 134–165.

In this paper the author develops the theory of Borel fields in certain group contexts to which he is led by his general study of the representation theory of separable locally compact groups. Some of his results are reformulations of the basic theory as presented by Kuratowski [Topologie, v. I, 2nd ed., Warsaw, 1948; MR **10**, 389], and some are wholly new consequences of applying this theory in the group situation.

It follows from Kuratowski's work that all non-countable Borel subsets of complete separable metric spaces are Borel isomorphic, and any σ-field Borel isomorphic to this common structure is called a standard Borel structure. Such a σ-field is countably generated, and has the crucial property (P) that any countable separating subfamily generates the whole σ-field. The range of a Borel measurable function on a standard domain may not be standard, even when it is countably generated; it then represents the next level of Borel complexity, and is called an analytic Borel structure in this paper. Repeated quotient processes lead no further: a Borel function having an analytic domain and a countably generated range has an analytic range. Analytic structures also have the crucial property (P). Moreover, from the point of view of measure theory the difference between an analytic structure and a standard one is often inconsequential, for a σ-finite measure on an analytic structure always has a set of measure 0 on the complement of which the structure is standard. In general, a measure on a Borel structure (σ-field) is called standard if the above holds.

If a Borel space is also a group there is an obvious definition of Borel measurability of the group, and one can consider standard Borel groups and analytic Borel groups. One of the author's most interesting results is his sharpening of Weil's theorem about the inverse of Haar measure. He proves that an analytic Borel group that carries a left-invariant Borel measure is necessarily already locally-compact under a uniquely determined topology having for its Borel sets exactly those of the given Borel structure. In a similar way, the closed subgroups of a locally compact group are characterized in terms of the Borel structure of their quotient spaces.

Finally, the author applies these ideas to the space of all irreducible representations of a given locally-compact separable group G into a given separable Hilbert space. This space of representations is a standard Borel structure in a natural way, and the quotient space obtained by identifying equivalent representations and imposing the natural quotient Borel structure is called the dual $\hat{G}$ of G. The dual $\hat{G}$ is not analytic unless it is countably separated, in which case it is said to be smooth. It is metrically smooth if every finite Borel measure on $\hat{G}$ has a null set whose complement is countably separated — hence, if every finite Borel measure on $\hat{G}$ is standard. The author conjectures that metrical smoothness of $\hat{G}$ may by equivalent to G having only type I representations. In any case, he shows that under both assumptions a multiplicity-free representation of G corresponds to a measure class (or ideal of null sets) on $\hat{G}$ in exact similarity to, and generalization of, the familiar characterization of a simple self-adjoint operator (or its equivalent one-parameter group of unitary transformations) by a Borel measure on the dual group $(-\infty, \infty)$. This generalization of the Hahn-Hellinger theory is necessary for the more refined aspects of the theory of the direct integral decomposition of reducible representations. *L. H. Loomis.*

Price, J. J. Some duality theorems. Illinois J. Math. **1** (1957), 433–445.

Let F be a countable family of functions $f_1, f_2, \cdots$ defined on a set S. The author calls F a Kronecker system if each f_i maps S to some group G_i of finite-dimensional complex matrices such that the complex conjugate $\bar{f}$ of any $f\in F$ is again in F and such that the Kronecker product $f_i\times f_j$ is up to a coordinate transformation a linear combination of the f_k. Defining the dual M of F as was done by T. Tannaka [Tôhoku Math. J. **45** (1938), 1–12] it follows easily that M is a compact topological group provided each G_i is compact. The author investigates such problems as that of finding conditions under which Kronecker systems can be regarded as representations of the dual groups. *F. I. Mautner* (Baltimore, Md.).

Lie Groups and Algebras

Mostow, G. D. Errata, "Factor spaces of solvable groups." Ann. of Math. (2) **66** (1957), 590.

The title paper appears in same Ann. (2) **60** (1954), 1–27 [MR **15**, 853]. These errata do not essentially invalidate the author's results.

Horie, Nobuo. On the group-space of the continuous transformation group with a Riemannian metric. Mem. Coll. Sci. Univ. Kyoto. Ser. A. Math. **30** (1956), 23–42.

It is shown that the group-space $S^{(R)}$ of any transformation group (not necessarily semi-simple) can be treated as a Riemannian space under the transformations of the parameters by defining the fundamental tensor by $\sum_{\alpha} A_{\alpha}{}^{a} A_{\beta}{}^{a}$. Moreover, the group-space $S^{(R)}$ is Euclidean when and only when the group is Abelian; and at every point in $S^{(R)}$ the scalar curvature is constant and can be made positive by a suitable choice of the $A_{\alpha}{}^{a}$'s when the group is non-Abelian and integrable. Finally, as an example the group-space $S^{(R)}$ of the group of linear substitutions $x'=ax+b$ $(a>0)$ is discussed in details.

C. C. Hsiung (Bethlehem, Pa.).

Poncet, Jean. Groupes de Lie compacts de transformations. C. R. Acad. Sci. Paris **245** (1957), 13–15.

The following results are announced. If E is a finite-dimensional compact metric space on which a compact group G operates effectively such that there are only a finite number of orbit classes, then there is an equivariant imbedding of E into some euclidean space on which G operates as a linear group. If a compact connected G operates effectively on some euclidean space E_n and if there is at least one $(n-1)$-dimensional orbit, then after suitable recoordinatization G is linear. If G is isomorphic to $SO(n-1)$ and if G operates on E_n such that there is an $(n-2)$-dimensional orbit, then again G operates linearly.

The first result has been proved by Mostow [Ann. of Math. (2) **65** (1957), 432–446; MR **19**, 291].

W. T. van Est (Leiden).

Laugwitz, Detlef. Über unendliche kontinuierliche Gruppen. II. Strukturtheorie lokal-Banachscher Gruppen. Bayer. Akad. Wiss. Math.-Nat. Kl. S.-B. **1956**, 261–286 (1957).

In this paper the author continues his study of infinite continuous groups, as begun in an earlier paper [Math. Ann. **130** (1955), 337–350; MR **17**, 762]. The objects of study are B-locally differentiable group-germs and their associated B-Lie algebras. By the latter is meant an abstract Lie algebra which is also a Banach space, with the product operation of the algebra being continuous. By means of Fréchet differentials and Taylor developments with remainder, a B-Lie algebra is associated with a differentiable group-germ.

It is proved that every B-Lie algebra is the result of such an association, and that the algebra determines the corresponding group-germ. This appears to the reviewer to be much the same as an old result of G. Birkhoff [Trans. Amer. Math. Soc. **43** (1938), 61–101], though the methods are different. Laugwitz utilizes an existence theorem of Michal and Elconin [Acta Math. **68** (1937), 71–107] on completely integrable differential equations in Banach spaces. There are also theorems on subgroups and the corresponding subalgebras. Let C be a closed subalgebra of the B-Lie algebra associated with a given group-germ K. Then there is a unique differentiable subgroup-germ whose associated B-Lie algebra is isomorphic to C. From this it can be proved that a subgroup-germ of a given B-locally differentiable group-germ K is differentiable, and differentiably imbedded in K, provided it is closed, has no nontrivial closed invariant subgroups, and contains an arc through the origin with a nonzero tangent vector there. The paper closes with some criteria for such a subgroup to be itself an invariant subgroup .

A. E. Taylor (Los Angeles, Calif.).

See also: Algebras: Taft. **Topological Groups:** Jacoby. **Differential Geometry:** Liber.

Topological Vector Spaces

Nachbin, Leopoldo. A generalization of Whitney's theorem on ideals of differentiable functions. Proc. Nat. Acad. Sci. U.S.A. **43** (1957), 935–937.

L'auteur donne un théorème contenant comme cas particulier le théorème de Hahn-Banach, et le théorème de Whitney sur les idéaux de fonctions différentiables [Amer. J. Math. **70** (1948), 635–658; MR **10**, 126]. Soit E un espace vectoriel topologique sur le corps des réels, $\mathscr{S}$ un ensemble de sous-espaces vectoriels de E; $X \subset E$ sera dit $\mathscr{S}$-convexe s'il est convexe et intersection des $X+S$, pour $S \in \mathscr{S}$; E est $\mathscr{S}$-localement convexe s'il a un système fondamental de voisinages de 0 qui sont $\mathscr{S}$-convexes. Soit A une algèbre commutative d'opérateurs continus de E, et soit $\mathscr{I}$ l'ensemble des idéaux I de A tels que A/I ait une dimension $\leq k$ et possède un seul idéal maximal, de codimension 1. Pour tout $I \in \mathscr{I}$, soit $I(E)$ la réunion des $T(E)$, pour $T \in I$; soit $\mathscr{S}(A)$ l'ensemble des sous-espaces vectoriels $I(E)$, pour $I \in \mathscr{I}$.

Moyennant deux conditions relatives à E et A, (1) exprimant que E est $\mathscr{S}(A)$-localement convexe, (2) exprimant une certaine condition de continuité pour les opérateurs de A, toujours vérifiée si E est normé, le théorème fondamental exprime que la "synthèse cospectrale" est possible pour E et A: tout sous-espace fermé de E, invariant par A, est l'intersection des sous-espaces "primaires" qui le contiennent. Un contre-exemple, tiré d'un théorème de Pollard sur l'approximation au sens de Bernstein des fonctions continues sur l'axe réel, montre que certaines des conditions imposées sont indispensables.

L. Schwartz (Paris).

Słowikowski, W. On a certain subclass of (DF) linear locally convex spaces. Bull. Acad. Polon. Sci. Cl. III. **5** (1957), 387–388, XXXI. (Russian summary)

The author defines a DB_0 space X as a linear locally-convex bornological space for which there is a sequence of Banach spaces $\{X_n\}$ each of which can be mapped linearly into X with no kernel and such that if B is a bounded subset of X then B is bounded in at least one X_n. The author has given a definition of strongest dual of a locally convex linear topological space (l.t.s.) [same Bull. **5** (1957), 113–115; MR **19**, 154] and he asserts the strongest dual of a metrizable locally convex l.t.s. is a DB_0 space. The DB_0 spaces satisfy property P defined in the review below.

D. G. Bourgin (Paris).

Słowikowski, W. (hF)-spaces and the Banach inversion property. Bull. Acad. Polon. Sci. Cl. III. **5** (1957), 487–489, XLI. (Russian summary)

A class K of real linear spaces has the Banach inversion property, P, if for every pair X, $Y \in K$ and every 1-1 linear transformation h of X onto Y, h^{-1} is continuous. A linear space, X, is an hF space if it is the union of a fixed ascending sequence of F spaces $\{X_m\}$ each with an F norm metric $\|\cdot\|_m$ (i.e., norm except for positive homogeneity), where $\|\{x_n|X_m\}\|_m \to 0$, $n \to \infty$, implies $\|\{x_n|X_m\}\|_{m+1} \to 0$. The topology of X induced by $\{X_n, \|\cdot\|_n\}$ is termed the hF

topology. The convergence notions are introduced in the usual way: $\{x_n\}\subset X$ is h-convergent to x_0, i.e., $x_n \xrightarrow{h} x_0$, if for some m, $\{x_n\}\cup(x_0)\subset X_m$ and $\|x_n-x_0\|_m\to 0$. Also $x_n \xrightarrow{h^*} x_0$ if for an arbitrary subsequence $\{x_n'\}$ and for some $\{x_{n_k}'\}\subset\{x_n'\}$, $x_{n_k}' \xrightarrow{h} x_0$; so h- or h^*-continuity of a distributive operation, $U: X\to Y$ is clear. The author asserts h-, h^*-continuity and the closure of the graph of U are all equivalent and that the hF spaces satisfy P. Extension is made to locally convex hF spaces, i.e., hF spaces admitting a bornological locally convex topology compatible in a natural way with the hF topology. *D. G. Bourgin.*

Amemiya, Ichiro. Some examples of (F) and (DF) spaces. Proc. Japan Acad. **33** (1957), 169–171.

This is a commentary on some unsettled questions in a paper by A. Grothendieck [Summa Brasil. Math. **3** (1954), 57–123; MR **17**, 765]. An (F) space E is said to have stability of boundedness if for every bounded set B in E and every dense subspace E_0 of E there exists a bounded subset A in E_0, with the closure of A containing B. It is shown by example that not all (F) spaces have this property. Next, there is given an (F) space which is reflexive, and yet such that its strong dual contains a bounded set for which the strong topology is not metrisable. Finally, there is an example of a bornological (DF) space with non-bornological bidual. *A. E. Taylor.*

Raikov, D. A. Inductive and projective limits with completely continuous mappings. Dokl. Akad. Nauk SSSR (N.S.) **113** (1957), 984–986. (Russian)

The author discusses (without proofs) inductive and projective (that is, direct and inverse) limit spaces of spectra of locally convex spaces under the extra hypothesis that the mappings of the spectrum are completely continuous. He generalizes results of Sebastião e Silva [Rend. Mat. e Appl. (5) **14** (1955), 388–410; MR **16**, 1122] and Grothendieck [Mem. Amer. Math. Soc. no. **16** (1955), Th. B; MR **17**, 763]. We quote the final theorem. Suppose that E is the projective limit of an inverse spectrum $(E_\alpha, \pi_\alpha{}^\beta)$, where the E_α are locally convex spaces, and for each pair of indices $\beta>\alpha$, $\pi_\alpha{}^\beta$ is a completely continuous linear mapping of E_β into E_α. Then: (1) E is a projective limit of an inverse spectrum of Banach spaces, and therefore is a complete locally convex space. (2) Each closed bounded set in E is compact. (3) E', taken with the strong topology, is isomorphic to the inductive limit of the direct spectrum $(E^\alpha, \pi^\alpha{}_\beta)$, where E^α is the strong dual of E_α and $\pi^\alpha{}_\beta$ is the mapping of E^α into E^β which is dual to $\pi_\alpha{}^\beta$ (under the assumption that $\pi_\alpha E$ is dense in E_α for each α). *M. M. Day* (Urbana, Ill.).

Bogdanov, Yu. S. Liapunov's norms in linear spaces. Dokl. Akad. Nauk SSSR (N.S.) **113** (1957), 255–257. (Russian)

Let A be a linear set, Δ an ordered set, and let $\lambda: A\to\Delta$ satisfy for any $a, a'\in A$ and any real c the conditions $\lambda(ca)\underline{=}\lambda(a)$, $\lambda(a+a')\underline{=}\max(\lambda a, \lambda a')$. The author calls λa the Lyapunov norm of a. He defines the notion of λ-basis in an n-dimensional subspace of A and states various properties of such bases. As an example he cites the space of solutions of systems of ordinary linear differential equations for which the Perron order numbers of solutions are a Lyapunov norm [cf., also, Markus, Math. Z. **62** (1955), 310–319; MR **17**, 37]. *H. A. Antosiewicz.*

Słowikowski, Wojciech. Extension de Schwartz des espaces vectoriels localement convexes. C. R. Acad. Sci. Paris **244** (1957), 2765–2767.

Généralisation abstraite des distributions. X étant un espace vectoriel topologique localement convexe, S un semigroupe d'opérateurs linéaires de sous-espaces de X dans X, on définit, moyennant des conditions convenables, un espace vectoriel topologique localement convexe PX, dans lequel X est plongé, et tel que les éléments de S opèrent linéairement et continuement de PX dans PX. *L. Schwartz* (Paris).

König, Heinz. Multiplikationstheorie der verallgemeinerten Distributionen. Bayer Akad. Wiss. Math.-Nat. Kl. Abh. (N.F.) no. **82** (1957), 80 pp.

On sait qu'il n'y a pas de théorie possible d'une multiplication des distributions. L'auteur appelle ici théorie de la multiplication la donnée d'un faisceau P d'espaces vectoriels P_x sur la droite réelle R, contenant comme sous-faisceau le faisceau D' des germes de distributions, d'une opération de dérivation, et d'une opération bilinéaire appelée multiplication de $D'_x\times P_x$ dans P_x, vérifiant la règle de dérivation du produit. On exige d'autre part que la dérivation et la multiplication coincident avec les opérations usuelles quand celles-ci existent, et que P soit engendré par D' multiplicativement. Abstraction faite du point de vue "faisceaux" ou passage du local au global, une théorie de la multiplication s'obtient en prenant le quotient de l'algèbre tensorielle de D' par un idéal à gauche $C(P)$, invariant par dérivation, contenant les $f\otimes g-fg$ pour les fonctions, et ne contenant aucun élément de D'. Le théorème 2 du § 6, p. 53, démontre l'existence de tels idéaux $C(P)$, donc de théories de la multiplication. Il existe alors un idéal minimal $C(P)$, donc une théorie maximale de la multiplication. La théorie maximale avait été étudiée dans un précédent mémoire [Math. Ann., **128** (1955), 420–452; MR **16**, 935]; ce mémoire donnait donc une démonstration du théorème d'existence, mais seulement pour la théorie maximale. Dans cette théorie maximale, le produit ST de deux distributions n'est lui-même une distribution que s'il a un sens par les procédés triviaux usuels; ainsi $\delta\delta$, δ vp $1/x$ ne sont pas des distributions. L'auteur signale ici une différence essentielle entre les théories de la dérivation et celles de la multiplication. On souhaite non seulement que toute fonction soit dérivable, mais aussi que seules des fonctions continues aient pour dérivées des fonctions; ceci amène à utiliser la théorie maximale de la dérivation, qui est celle des distributions. Ici on souhaite bien que deux distributions aient un produit, mais aussi que le plus souvent possible ce produit soit lui-même une distribution; aussi la théorie maximale n'est-elle pas spécialement intéressante. L'auteur construit des théories-standard, différentes de la maximale, en infinité, ayant une puissance supérieure à celle du continu, deux à deux non isomorphes. Dans chacune d'elles, le produit ST de deux distributions est une distribution, toutes les fois que T n'a que des singularités isolées. Par example δ vp $1/x$ vaut $-\delta'+c\delta$, où c dépend de la théorie.

Cet article est long, mais minutieusement rédigé, et il contient tous les développements axiomatiques conduisant logiquement aux diverses théories possibles. L'abondance des multiplications pouvant être utilisées est un obstacle assez sérieux pour les applications éventuelles. Mais d'un autre côté certaines considérations supplémentaires d'invariance (parmi lesquelles par exemple l'invariance de Lorentz) sont peut-être susceptibles

de conduire, en théorie quantique des champs, à une théorie de multiplication privilégiée parmi d'autres, s'appuyant sur le présent formalisme. *L. Schwartz.*

Mirakov, V. E. The majorant principle and the method of tangent parabolas for non-linear functional equations. Dokl. Akad. Nauk SSSR (N.S.) **113** (1957), 977–979. (Russian)

The problem is considered of finding an approximate solution of the functional equation $P(x)=0$, where $y=P(x)$ is a non-linear operator mapping $x \in X$ onto $y \in Y$. Here X and Y are complete linear spaces, of type B_k, normed according to L. V. Kantorovič by means of semi-ordered spaces Z and W [Dokl. Akad. Nauk SSSR (N.S.) **76** (1951); MR **12**, 835]. *From the introduction.*

Ishihara, Tadashige. Divergent integrals as viewed from the theory of functional analysis. I. Proc. Japan Acad. **33** (1957), 92–97.

Let $v(\sigma, \tau, s)$ be a continuous function of the 3 real variables σ, τ, s for $0 \leqq s < \infty$; we call $z=\sigma+i\tau$. Call $D(\tau)$ the space of C^∞ functions of compact support, and denote by $Z(\sigma)$ the Fourier transform of $D(\tau)$, and call Φ the projective tensor product of $Z(\sigma)$ with $D(\tau)$. We assume that for each $\varphi \in \Phi$ and each s we have $v\varphi \in \Phi$. Then, under certain other conditions, the author defines $v^*(z)=\int_0^\infty \exp(izs)v(\sigma, \tau, s)ds$ as an element of Φ' (dual of Φ) by the formula

$$v^*(z)\cdot\varphi(z)=\int_0^\infty ds\int_{-\infty}^{\infty} d\sigma\int_{-\infty}^{\infty} d\tau \exp(izs)v(\sigma, \tau, s)\varphi(\sigma, \tau).$$

In case v is independent of σ, τ, the method is due to the reviewer [Ann. of Math. (2) **63** (1956), 129–159; MR **17**, 876]. *L. Ehrenpreis* (Waltham, Mass.).

Trèves, François. Sur les correspondances vectorielles. C. R. Acad. Sci. Paris **245** (1957), 1200–1203.

Let E be a vector space (no topology) and F a normed vector space. We denote by $L(E; F)$ the space of linear maps of E into F and by $\mathrm{Aut}(F)$ the group of bounded and invertible linear maps of F onto itself. For $S, T \in L(E; F)$ we say that S dominates T if for each $\varepsilon>0$ we can find an $A \in \mathrm{Aut}(F)$ such that $\|A(Tx)\| \leqq \varepsilon\|A(Sx)\|$ for each $x \in E$. Th. 1: If S dominates T, then we can find a constant C so that $\|Sx\| \leqq C\|(S+T)x\|$ for $x \in E$.

The author applies his methods to partial differential operators on Euclidean space and, by domination over constant coefficient operators, the author states that he can prove invertibility properties of variable coefficient equations. *L. Ehrenpreis* (Waltham, Mass.).

Trèves, François. Sur les correspondances vectorielles. C. R. Acad. Sci. Paris **245** (1957), 1288–1291.

Let $P(D)$ be a partial differential operator with constant coefficients on R^n. The author gives necessary and sufficient conditions that $P(D)$ dominate the constant 1. Using his notion of domination, the author states that if $P(D)$ is hypoelliptic (all distribution solutions of $P(D)f=0$ are C^∞ functions) and if $P(D)$ equidominates $Q_j(D)$ $(j=1, 2, \cdots, n)$, then $P(D)+\sum a_j(x)Q_j(D)$ is hypoelliptic for any C^∞ functions a_j. *L. Ehrenpreis.*

See also: Trigonometric Series and Integrals: Konyušков. **Ordinary Differential Equations:** Mikusiński. **Partial Differential Equations:** Lions. **Banach Spaces, Banach Algebras, Hilbert Spaces:** Sudakov; Love; Smart; Rosenblum; Foiaş; Gussi and Poenaru. **Numerical Methods:** Altman.

Banach Spaces, Banach Algebras, Hilbert Spaces

Borisovič, Yu. G. On critical values of certain functionals in Banach spaces. Uspehi Mat. Nauk (N.S.) **12** (1957), no. 1(73), 157–160. (Russian)

The author generalizes the definition of genus, introduced by Krasnoselskiĭ [Uspehi Mat. Nauk (N.S.) **7** (1952), no. 2(48), 157–164; MR **14**, 55 (in this review the reviewer translated the Russian word "rod" as "kind" rather than "genus")]. Let R be a metric space and A a continuous mapping of R into itself. Let E be a closed set in R, not containing any fixed point of A. E has genus 1 with respect to A if no connected component of E contains a pair $(\phi, A\phi)$. E has genus n if it is included in the union of n closed sets of genus 1 and not in the union of $n-1$ such sets. As did Krasnoselskiĭ's genus, the author's genus possesses many of the properties of category. He applies it to an example, as follows. Let R be a linear normed space. Let A be a unitary operator if R is a Hilbert space, or an involution if R is a Banach space. Let F and Φ be functionals on this space, with $F(A\phi)=F(\phi)$ and $\Phi(A\phi)=\Phi(\phi)$, and F weakly continuous. ϕ and ψ are said to be essentially different if neither can be gotten from the other by repeated applications of A. A critical point of F on the surface $\Phi(\phi)=k$ is a point at which $\mathrm{grad}\, F(\phi)-\lambda\, \mathrm{grad}\, \Phi(\phi)=0$. By constructing a new deformation operator, the author proves that the number of essentially different critical points of F on the surface $\Phi(\phi)=k$ is not less than the genus of that surface with respect to A. *J. M. Danskin* (Princeton, N.J.).

Ruston, A. F. Conjugate Banach spaces. Proc. Cambridge Philos. Soc. **53** (1957), 576–580.

Two necessary and sufficient conditions for a Banach space to be isomorphic to a conjugate space are proved: (1) existence of a minimal total closed subspace of its conjugate space; (2) existence of a projection of its second conjugate space onto the canonical image which annihilates a regularly closed subspace. The isomorphism is an equivalence if the subspace of functions is duxial (every element induces a functional of equal norm in the conjugate of the subspace), or the projection has norm one. These results are equivalent to theorems by Dixmier [Duke Math. J. **15** (1948), 1057–1071; MR **10**, 306]. *D. C. Kleinecke* (Livermore, Calif.).

James, Robert C. Reflexivity and the supremum of linear functionals. Ann. of Math. (2) **66** (1957), 159–169.

If B is a real Banach space with unit sphere S, then reflexivity of B implies compactness of fS for each $f \in B^*$. A long-standing question asks whether the reverse implication holds [Mazur, Studia Math. **4** (1933), 128–133]. Although some partial results were obtained in 1950 by the author [Bull. Amer. Math. Soc. **56** (1950), 58] and the reviewer [Rev. Ci., Lima **52** (1950), nos. 3–4, 15–23; MR **13**, 250], the present paper contains the first significant contribution to the problem. Its principal result asserts that if a Banach space B has a complemented nonreflexive subspace, then B admits a continuous linear functional not attaining its supremum on the unit sphere of B. This implies an affirmative answer to Mazur's question for separable Banach spaces, and shows also that an arbitrary Banach space is reflexive if and only if each continuous linear functional attains a maximum on the unit sphere of each separable closed subspace. Some other consequences are obtained, including extensions of the

reviewer's geometric characterizations of reflexivity. The author's basic tool is the following lemma, which is itself based on the Šmulian-Eberlein characterization of reflexive spaces as those in which a decreasing sequence of nonempty bounded closed convex sets must have nonempty intersection:

Lemma 1. Let B be a Banach space which is the closed linear span of a sequence $\{x_i\}$. Then B is non-reflexive if and only if there is a positive number σ such that, for any positive numbers Δ and δ with $\Delta>\delta$, there exist elements $\{z_i\}$ for which: (i) $\Delta(\sum_1 \alpha_i)\geqq\|\sum_1 \alpha_i z_i\|\geqq\delta(\sum_1 \alpha_i)$ for all finite sequences of non-negative numbers $\{\alpha_i\}$; (ii) For any n, there is an $N(n)$ such that, for any numbers $\{a_i\}$ and any non-negative numbers $\{\beta_i\}$,

$$\textstyle\sum_1^n a_i x_i+\sum_N \beta_i z_i\|\geqq\sigma\|\sum_N \beta_i z_i\|.$$

V. Klee (Seattle, Wash.).

Civin, Paul; and Yood, Bertram. Quasi-reflexive spaces. Proc. Amer. Math. Soc. **8** (1957), 906–911.

A Banach space X is called quasi-reflexive of order n if X^{**}/X is n-dimensional. It is shown that if E is a closed linear manifold in X, then X is quasi-reflexive of order n if and only if E and X/E are so of orders m and p respectively and $n=m+p$. Any weakly complete quasi-reflexive space is reflexive; every non-separable quasi-reflexive space has a non-separable reflexive subspace.

I. Halperin (Kingston, Ont.).

Schatten, Robert. The space of completely continuous operators on a Hilbert space. Math. Ann. **134** (1957), 47–49.

Let B be a Banach space. The author observes that if for each x in B, with $\|x\|\leq 1$, there is a non-zero y in B with $\|x\pm y\|\leq 1$, then by a corollary to the Krein-Milman theorem [Studia Math. **9** (1940), 133–138; MR **3**, 90] B cannot be the conjugate of a Banach space. He shows that the space of completely continuous linear operators on a Hilbert space (infinite-dimensional) satisfies the above condition.

J. H. Williamson (Belfast).

Long, Richard G. A T-system which is not a Bernstein system. Proc. Amer. Math. Soc. **8** (1957), 925–927.

Let u_i $(1\leq i\leq\gamma)$ be a linearly independent fundamental set in the (finite-dimensional or separable) Banach space E; let L_n, for $n\leq\gamma$, be the linear subspace spanned by the u_i, $1\leq i\leq n$; $\{u_i\}$ is a T-system if for each x in E and each $n\leq\gamma$ there is a unique point $P_n(x)=\sum_{i\leq n} x_{i,n}u_i$ of L_n closest to x. For a given T-sequence in E a deviation sequence is defined for each x in E; the T-system is called a Bernstein system if x is determined by its deviation sequence. This note answers in the negative a question of Kadec [Dokl. Akad. Nauk SSSR (N.S.) **92** (1953), 465–468; MR **15**, 535] by displaying in a space E isomorphic to c_0 a T-system which is not a Bernstein system.

M. M. Day (Urbana, Ill.).

Sudakov, V. N. Criteria of compactness in function spaces. Uspehi Mat. Nauk (N.S.) **12** (1957), no. 3(75), 221–224. (Russian)

The author shows that the boundedness hypothesis in some well-known criteria for compactness in L^p spaces [Kolmogoroff, Nachr. Ges. Wiss. Göttingen. Math.-Phys. Kl. **1931**, 60–63; Tamarkin, Bull. Amer. Math. Soc. **38** (1932), 79–84; Krasnosel'skiĭ and Rutitskiĭ, Dokl. Akad. Nauk SSSR (N.S.) **85** (1952), 33–36; MR **14**, 57] is superfluous.

M. M. Day (Urbana, Ill.).

Love, E. R. A Banach space of distributions. I. J. London Math. Soc. **32** (1957), 483–498.

L'auteur définit l'espace de Banach des distributions portées par l'intervalle réel $(0, c)$, dont une primitive $(k+1)$-ième est une fonction sommable sur $(0, c)$, comme le complété de l'espace des fonctions continues pour la norme

$$\int_0^c dx\left|\int_0^x f(t)(x-t)^k dt\right|.$$

L. Schwartz (Paris).

Smart, D. R. Relations between the spectrum of an infinite matrix and the spectra of its sections. J. London Math. Soc. **32** (1957), 357–367.

Let H be an infinite matrix, $H=(h_{ij})$ and H_n its nth section, $H_n=(h_{ij})$ $(1\leq i, j\leq n)$. Let γ be the set of complex numbers obtained from the sequence of spectra of the H_n by some limit process. In this paper the author studies the problem of establishing relationships between γ and the spectrum of H.

F. I. Mautner (Baltimore, Md.).

Gavurin, M. K. Approximate determination of eigenvalues and the theory of perturbation. Uspehi Mat. Nauk (N.S.) **12** (1957), no. 1(73), 173–175. (Russian)

The article states, without derivation, a few bounds for characteristic values of perturbed operators. The principal result appears to be: Let A, A_0, B be self-adjoint linear operators such that $A=A_0+B$. Let λ_0 be a simple characteristic value of A_0, λ the nearest characteristic value of A. Let ϕ_0, ϕ be the corresponding characteristic vectors, $\rho=|\lambda_0-\lambda|$, $s=\|B\|/\rho<\frac{1}{2}$, and "normalize" ϕ by requiring $(\phi, \phi_0)=1$. Then

$$|\lambda-(A\phi_0, \phi_0)|\leq\|\phi-\phi_0\|\,\|B\phi_0\|,\ \|\phi_0-\phi\|\leq\frac{2s}{\sqrt{(1-4s^2)}}.$$

R. E. Gaskell (Seattle, Wash.).

Boigelot, A. Le théorème spectral pour les opérateurs normaux bornés d'un espace de Hilbert dans lui-même. Bull. Soc. Roy. Sci. Liège **26** (1957), 123–130.

Let N be a bounded normal operator on a Hilbert space H. The author gives a proof of the spectral theorem; i.e., if $\varphi(\lambda, \mu)$ is a continuous function of two complex variables defined on the spectrum S of N, then there exists a spectral measure E defined on S such that $\varphi(N, N^*)=\int_S \varphi(\xi, \bar{\xi})d_\xi E$. The proof makes use of the theorem of F. Riesz on representations of linear functionals defined on a space of continuous functions. The method used is almost identical with that used by Eberlein in the case of a self-adjoint operator [Bull. Amer. Math. Soc. **52** (1946), 328–331; MR **7**, 453]. The author seems to be unaware of Eberlein's paper since it is not quoted.

R. E. Fullerton (College Park, Md.).

Rosenblum, Marvin. Perturbation of the continuous spectrum and unitary equivalence. Pacific J. Math. **7** (1957), 997–1010.

Let A and B be selfadjoint operators on a Hilbert space H such that $P=B-A$ is a completely continuous operator. The author investigates additional conditions on A and B which insure that they be unitarily equivalent. Specifically, assume that the following conditions are satisfied: (a) The trace of $|P|$ is finite. (b) If $A=\int_{-\infty}^{\infty} x\,d_x E_x$, $B=\int_{-\infty}^{\infty} x\,d_x F_x$, then $(E_x f, g)$, $(F_x f, g)$ are absolutely continuous functions of x for every $f, g\in H$. Then it is shown that $\lim_{t\to\infty} e^{-iBt}e^{iAt}=U$, $\lim_{t\to-\infty} e^{-iBt}e^{iAt}=V$ strongly, U and V are unitary and $U^*BU=A$, $V^*AV=B$.

R. E. Fullerton (College Park, Md.).

Foiaş, Ciprian. Sur certains théorèmes de J. von Neumann concernant les ensembles spectraux. Acta Sci. Math. Szeged **18** (1957), 15–20.

Let A be a Banach algebra with unit and involution. Then, following von Neumann, a set S of complex numbers is said to be a spectral set of an element x of A if, for every rational r with $|r(\lambda)|\leqq 1$ for λ in S, the element $r(x)$ exists in A and has $|r(x)|\leqq 1$. It is shown that if the half-plane $\mathrm{Re}\,\lambda\geqq 0$ is a spectral set of x then $\mathrm{Re}\,f(x)\geqq 0$ for all linear positive forms f on A. If, conversely, A has the property that the half-plane $\mathrm{Re}\,\lambda\geqq 0$ is a spectral set of x provided that $\mathrm{Re}\,f(x)\geqq 0$ for every linear positive form f on A, then A is isometrically isomorphic with a closed subalgebra of the algebra of bounded linear operators in a Hilbert space. It is shown (as was also known to von Neumann) that a complex B-space X is necessarily a Hilbert space if the unit disk $|\lambda|\leqq 1$ is a spectral set of every operator T in X with $|T|\leqq 1$. *N. Dunford.*

Foiaş, C.; Gussi, G.; and Poeńaru, V. On the basic approximation theorem for semigroups of linear operators. Proc. Nat. Acad. Sci. U.S.A. **43** (1957), 616–618.

This paper contains a generalization of the basic approximation theorem for semigroups of linear bounded operators on a (B)-space X due to E. Hille [Th. 9.3.4, Functional analysis and semi-groups, Amer. Math. Soc. Colloq. Publ., vol. 31, New York, 1948; MR **9**, 594]. The result extends a similar theorem due to B. Sz.-Nagy for the case where the semigroup is strongly continuous to the identity at the origin [see p. 396 of F. Riesz and B. Sz.-Nagy, Leçons d'analyse fonctionnelle, 2ème éd., Akad. Kiadó, Budapest, 1953; MR **15**, 132]. The theorem reads: Given a semi-group $[T(s)]$ of bounded linear operators on X, strongly continuous for $s>0$, and given a family of bounded linear operators B_ε, $0<\varepsilon<1$, such that (i) B_ε commutes with $T(s)$; (ii) $\|[\exp(hB_\varepsilon)]T(t_0)\|\leqq M(t_0, t_1)$ for $0\leqq h\leqq t_1-t_0$; (iii) $B_\varepsilon x\to Ax$ as $\varepsilon\to 0+$ for each x in $D(A)$, where A is the infinitesimal generator of $[T(s)]$. Then, for each $x\in X$, $T(s)x=\lim_{\varepsilon\to 0+}\exp[(s-\alpha)B_\varepsilon]T(\alpha)x$. Here $\alpha>0$, $s>\alpha$, and the convergence is uniform in s, $\alpha\leqq s\leqq\beta<\infty$. The reviewer caught one typographical error, namely, a factor of 'n' is missing in the right member of the last inequality on page 616. *R. S. Phillips.*

Zaidman, Samuel. Sur la représentation des fonctions vectorielles par des intégrales de Laplace-Stieltjés. C. R. Acad. Sci. Paris **245** (1957), 397–399.

The author derives necessary and sufficient conditions for the representation of an abstract-valued function as a Laplace-Stieltjes integral. [For Laplace integrals, see P. G. Rooney, Canad. J. Math. **6** (1954), 190–209; MR **15**, 620; and I. Miyadera, Tôhoku Math. J. (2) **8** (1956), 170–180; MR **18**, 748.] Let V_f, V_r, V_G be the classes of functions $\alpha(t)$ on $[0, \infty]$ to the B-space X which are of bounded variation on $[0, \infty]$ in the weak, strong, and Gel'fand sense, respectively. If X is reflexive and $\alpha(t)\in V_f$ and if $f(s)=\int_0^\infty e^{-st}d\alpha(t)$, then

$$\alpha(t)-\alpha(0+)=\lim_{k\to\infty}\alpha_k(t)=\lim_{k\to\infty}\int_{0+}^{t}L_{k,\tau}[f(\cdot)]d\tau,$$

where

$$L_{k,\tau}[f(\cdot)]=\frac{(-1)^k}{k!}\left(\frac{k}{\tau}\right)^{k+1}f^{(k)}\left(\frac{k}{\tau}\right)$$

is the Widder operator. Let the range of $\{\alpha_k(t)|\ 0\leqq t,\ k=1, 2, \cdots\}$ be $L_1\subset X$. Let L_2 denote the set of elements of the form $\sum_1^n \varepsilon_i[\alpha_k(t_{i+1})-\alpha_k(t_i)]$ with $\varepsilon_i=\pm 1$. An extension of the Helly theorem holds if L_1 and L_2 are relatively compact; then there exists a subsequence converging strongly pointwise to an $\alpha(t)\in V_G$. The author introduces three conditions: (A) $\int_0^\infty\|L_{k,\tau}[f(\cdot)]\|d\tau\leqq M$, ($A_1$) $\alpha_k(t)$ exists and L_2 is bounded in X, (A_2) $\alpha_k(t)$ exists and L_2 is relatively compact. If X is reflexive, then (A_1) is necessary and sufficient for the existence of a normalized $\alpha(t)\in V_f$ with $f(s)=\int_0^\infty e^{-st}d\alpha(t)$. If V_f is replaced by V_r, (A_1) is replaced by (A). In an arbitrary B-space, replace (A_1), V_f by (A_2), V_G or by (A)+(A_2), $V_G\cap V_r$. *E. Hille.*

See also: Measure, Integration: Kampé de Fériet. **Trigonometric Series and Integrals:** Konyuškov. **Ordinary Differential Equations:** Krasnoselsky, Krein and Sobolevsky, Krein. **Lie Groups and Algebras:** Laugwitz. **Topological Vector Spaces:** Słowikowski. **Numerical Methods:** Altman.

TOPOLOGY

General Topology

Šik, František. Einige Bemerkungen über die Topologien mit vorgeschriebenen Gestirnen der Punkte. Publ. Fac. Sci. Univ. Masaryk 1955, 473–480. (Czech. Russian and German summaries)

A continuation of the article in same Publ. 1955, 459–472 [MR **19**, 298]. A topology u on a set P is called an O-topology if the closure of the empty set is empty, an I-topology if $M\subset uM$, an A-topology if $u(M\cup N)=uM\cup uN$, an A^*-topology if $u\bigcup_\alpha M_\alpha=\bigcup_\alpha uM_\alpha$, and finally a B-topology if $u(x)=(x)$. The set of all points $x\in P$ for which $O_u(x)$ is not empty is denoted by P'. {A neighborhood of $M\subset P$ is any O such that $M\cap u(P\setminus O)$ is void; $O_u(M)$ is the set of neighborhoods of M; the "constellation" M^g of M is $O_u\cap(M)$.}

It is proved that a topology u is an M-topology (i.e., is monotone) if and only if an arbitrary set containing a neighborhood of a given point is itself a neighborhood of that point. A topology u is an I-topology if and only if an arbitrary point of the set P' is contained in its constellation. An M-topology u is 1) an A-topology if and only if from $x\in P'$ and O, $O'\in O_u(x)$ it follows that $O\cap O'\in O_u(x)$; 2) an A^*-topology if and only if from $x\in P'$, $\{O_\alpha\}\subset O_u(x)$ it follows that $\bigcap O_\alpha\in O_u(x)$; 3) an O-topology if and only if $P'=P$; 4) a B-topology if and only if $x^g=(x)$ for all $x\in P$. It is shown that the least element of the set $\mathfrak{P}$ (of all topologies with a given set of constellations of points) is an M- and an A^*-topology, and necessary and sufficient conditions are found for this topology to be an O-, an I- or a B-topology. A number of conditions are also derived for the existence in the set $\mathfrak{P}$ of a greatest element. *M. F. Bokštein* (RŽMat **1957**, no. 1218).

Iséki, Kiyoshi. A characterisation of pseudo-compact spaces. Proc. Japan Acad. **33** (1957), 320–322.

Let S be a completely regular T_0-space. A sequence $\{f_n\}_{n=1}^\infty$ of real-valued functions on S is said to converge uniformly to f at $x\in S$ if, for every $\varepsilon>0$, there are an index N and a neighborhood U of x such that $|f_n(y)-f(y)|<\varepsilon$ for all $n\geqq N$ and $y\in U$. Main theorem: S is pseudo-compact (i.e., all real-valued continuous functions are bounded) if and only if every sequence of continuous real functions on S that converges to a continuous function

uniformly at every point of S actually converges uniformly on all of S. Several similar results are obtained.

E. Hewitt (Seattle, Wash.).

Iséki, Kiyoshi. On weakly compact and countably compact topological spaces. Proc. Japan Acad. **33** (1957), 323–324.

This note continues earlier results by the same author [same Proc. **33** (1957), 252–254; MR **19**, 239]. Let S be a regular T_0-space. Theorem 1: S is feebly compact $\Leftrightarrow$ every point-finite countable open covering of S has a proper subfamily such that the union of the closures of the elements of the subfamily is S $\Leftrightarrow$ every point-finite countable open covering of S has a proper subfamily such that the closure of the union of the elements of the subfamily is S. Theorem 2: S is countably compact $\Leftrightarrow$ every point-finite infinite open covering of S has a proper subcovering $\Leftrightarrow$ every point-finite countable infinite open covering of S has a proper subcovering. *E. Hewitt.*

Hanai, Sitiro. On open mappings. Proc. Japan Acad. **33** (1957), 177–180.

Let $f(X)=Y$, where X and Y are topological spaces and f is continuous. (1) If f is both open and closed, and X is semicompact (=locally peripherally compact), then Y is semicompact. Two further theorems are partial extensions of a theorem of the reviewer [Proc. Amer. Math. Soc. **7** (1956), 690–700; MR **19**, 299] to non-metric spaces; both are included in the following more complete (and routine) extension. (2) If f is quasicompact, X is semicompact, Y is Hausdorff, and (for each $y \in Y$) $f^{-1}(y)$ is connected and has a compact frontier, then f is closed and Y is semicompact. An example is presented to show that (1) would be false if f were not assumed to be closed. {Though simple examples of this kind exist, the one given seems to the reviewer to be incorrect, the space Y being in fact semicompact.} A second example shows that (2) needs the condition that the sets $f^{-1}(y)$ have compact frontiers.

A. H. Stone (Manchester).

Morita, Kiiti. On closed mappings. II. Proc. Japan Acad. **33** (1957), 325–327.

[For part I see same Proc. **32** (1956), 539–543; MR **19**, 49.] The author's principal result is the following theorem, which had previously been obtained by A. H. Stone [Proc. Amer. Math. Soc. **7** (1956), 690–700; MR **19**, 299] for the special case of metrizable X, and by S. Hanai [see the paper reviewed above] for the (somewhat less) special case of first-countable Y: If f is a quasi-compact (=quotient) mapping from a locally peripherally compact Hausdorff space X onto a Hausdorff space Y, and if $f^{-1}(y)$ is connected and has compact boundary for every $y \in Y$, then f is closed. *E. Michael* (Seattle, Wash.).

Suzuki, Jingoro. Some properties of completely normal spaces. Proc. Japan Acad. **33** (1957), 19–24.

The author gives several necessary and sufficient conditions for a topological space to be completely normal. The following theorem, his main result, is representative. H is completely normal if and only if for any locally finite family $\{X_\alpha : \alpha \in \Omega\}$ of closed subsets of H there exists a family $\{H_\alpha : \alpha \in \Omega\}$ of closed subsets with the properties (i) $H=\bigcup H_\alpha$ and (ii) $H_\alpha \cap H_\beta \cap (X_\alpha \cup X_\beta) = X_\alpha \cap X_\beta$. For finite families this result is due to T. Inokuma [same Proc. **31** (1955), 56–59; MR **16**, 1041]. *H. H. Corson.*

Hayashi, Yoshiaki. On dimensions of topological spaces. Bull. Univ. Osaka Prefecture. Ser. A. **5** (1957), 1–4.

In a previous paper [Math. Japon. **3**, 71–84 (1954), 136 (1955); MR **16**, 1139] the author defines a dimension for arbitrary topological spaces which satisfies the monotone property, the sum theorems and the decomposition theorem, and which agrees with the Urysohn-Menger dimension for separable metric spaces. In the present paper he generalizes this function to a class of dimension functions each of which, he shows, satisfies the conditions above. *Haskell Cohen* (Baton Rouge, La.).

Papić, Pavle. Sur une classe d'ensembles ordonnés et les espaces pseudo-distanciés. Glasnik Mat.-Fiz. Astr. Ser. II. **11** (1956), 161–168. (Serbo-Croatian summary)

α, β étant nombres ordinaux, soit $\alpha(\beta)$ l'ensemble de toutes les β-suites d'ordinaux $<\alpha$. Soit $O_\alpha(\beta)$ l'espace ordonné induit par l'ordination alphabétique de $\alpha(\beta)$. Soit $B_\nu(\omega_\alpha)$ l'espace pseudo-distancié $\nu(\omega_\alpha)$, $\nu \leqq \omega_\alpha$, défini par la pseudo-métrique δ telle que $\delta(x,x)=\omega_\alpha$, $\delta(x,y)_{\neq}=\xi$, ξ étant le premier indice vérifiant $x_\xi \neq y_\xi$, où $x=(x_\varphi)$, $y=(y_\varphi)$ $(\varphi<\omega_\alpha)$. (Th. 1) L'espace $B_\nu(\omega_\alpha)$ est complet. (Th. 4) $O_\nu(\omega_\alpha)$ contient un espace pseudo-distancié partout dense. (Th. 6) Tout espace pseudo-distancié de la classe (D_α) et $\aleph_\alpha$-séparable est homéomorphe d'un sous-espace de $B_2(\omega_\alpha)$; si $\alpha>0$ et $E \in (D_\alpha)$, E est homéomorphe d'un sous-espace de $B_{\omega_\beta}(\omega_\alpha)$, pour un certain ordinal β.

Đ. Kurepa (Zagreb).

Wagner, K. Über nicht-archimedische Metrisierbarkeit in n-fach geordneten Mengen. Math. Ann. **134** (1957), 33–40.

A metric $\mathfrak{M}_{(+)}$ is a set $\mathfrak{M}$ with total ordering $<$ and composition $+$ (not necessarily commutative, associative) such that (1) order is preserved under left and right translations; (2) there exists an element O with $O \leqq \alpha \leqq O+\alpha$ for all $\alpha \in \mathfrak{M}$; and (3) there exists a sequence $\alpha_1, \alpha_2, \cdots$ such that $\alpha_n+\alpha_n$ converges to O. — Now let X be a totally ordered set in which subsets with upper bounds have least upper bounds. Then X has the order type of an interval in the system of real numbers provided there exists a metric $\mathfrak{M}_{(+)}$ and a mapping $\rho: X \times X \to \mathfrak{M}$ such that $\rho(x,y)$ satisfies the usual axioms for a metric (in the ordinary sense) and the system of neighborhoods of X, defined by means of the ordering, is equivalent to that defined by ρ. *P. A. Smith* (New York, N.Y.).

Dal Soglio, Letizia. Grado topologico e teoremi di esistenza di punti uniti per trasformazioni plurivalenti di n-celle. Rend. Sem. Mat. Univ. Padova **27** (1957), 103–121.

The author attaches a degree to certain types of transformations (not necessarily single-valued) acting on subsets of euclidean n-space and obtains fixed-point theorems. The results and methods are straightforward generalizations of those with which the author treated the case $n=3$ [same Rend. **25** (1956), 386–405; MR **18**, 141]. *P. A. Smith* (New York, N.Y.).

★ **Parchomenko, A. S. Was ist eine Kurve?** VEB Deutscher Verlag der Wissenschaften, Berlin, 1957. 142 pp. (1 insert) DM 8.20.

This book, translated by E. Mundry and W. Schmidt from the Russian (Gosudarstv. Izdat. Tehn.-Teor. Lit., Moscow 1954), is intended chiefly for future teachers of mathematics. The first chapter gives a short historical

sketch with emphasis on the need for point-set topology for a general definition of a curve; the second gives the necessary point-set material; the third gives the Cantor definition of a curve (satisfactory for plane curves); and the fourth gives the general Urysohn definition based on the dimension-theory (discussed in an appendix) of Menger-Urysohn. There is an index on a loose, inserted sheet.

See also: Banach Spaces, Banach Algebras, Hilbert Spaces: Borisovič. **Convex Domains, Integral Geometry:** Grünbaum.

Algebraic Topology

★ **Aleksandrov, P. S. Combinatorial topology. Vol. 2. The Betti groups.** Graylock Press, Rochester, N. Y., 1957. xi+244 pp. $6.50.

Translation by Horace Komm from the Russian Kombinatornaya topologiya, Part III, with the appendix on Abelian groups [Gostehizdat, Moscow, 1947; MR **10**, 55]. The translation of Parts I and II is listed in MR **17**, 882.

★ **Borel, Armand. Cohomologie des espaces localement compacts, d'après J. Leray. Séminaire de topologie algébrique, printemps 1951. 2me éd.** Ecole Polytechnique Fédérale, Zürich, 1957. 95 pp. (polycopiées) 7.00 francs suisses.

The second edition of these seminar notes follows closely the first edition [1953; MR **15**, 458] with the following changes. The definition of couverture in Chapter II is modified along lines suggested by Fary so that it refers directly to a complex with compact supports. The Lazard definition of sheaf is given in Chapter V, and this has led to a completely new treatment of the material of that chapter.

These changes have made possible substantial simplifications in the presentation and have brought the material up to date so that it is in agreement with current terminology. This modernization makes the notes even more useful as an introduction to the theories of cohomology of locally compact spaces, sheaves, and spectral sequences. *E. H. Spanier* (Chicago, Ill.).

★ **Adams, J. F. On the cobar construction.** Colloque de topologie algébrique, Louvain, 1956, pp. 81–87. Georges Thone, Liège; Masson & Cie, Paris, 1957. 375 fr. belges; 3000 fr. français.

The "co-bar" construction $F(C)$ for a graded associative co-algebra C with differentiation is defined. It is then shown that if C is the singular chain complex of a 1-connected topological space B then the homology groups of $F(C)$ are naturally isomorphic with those of the loop space of B. *S. Eilenberg* (New York, N.Y.).

Forrester, Amasa. Acyclic models and de Rham's theorem. Trans. Amer. Math. Soc. **85** (1957), 307–326.

In this paper the author abstracts and extends the theory of acyclic models which was originated by Eilenberg and MacLane [Amer. J. Math. **75** (1953), 189–199; MR **14**, 670]. The exposition is fairly clear except at one point. The reviewer is unable to understand the proof of either Theorem 9A or 9B.

Without going into a great number of technical details concerning categories and functors, it is difficult to describe the author's theory of acyclic models or how it generalizes that of Eilenberg and MacLane. However, using this theory it is possible to give a proof of de Rham's theorem. The proof is concise and conceptional and would not be possible in the original theory of acyclic models of Eilenberg and MacLane or in a weaker generalization developed by Gugenheim and the reviewer [Trans. Amer. Math. Soc. **85** (1957), 265–306; MR **19**, 160]. Unfortunately, however, the previously mentioned proof of de Rham's theorem depends strongly on Theorem 9B.

In addition to the preceding, the author gives a treatment of cup products by using acyclic models. This treatment is an appropriate generalization of the classical one using acyclic carriers, and has many technical advantages over the classical treatment. Its essential features do not depend on either Theorem 9A or 9B.

J. C. Moore (Princeton, N.J.).

Kan, Daniel M. On c. s. s. complexes. Amer. J. Math. **79** (1957), 449–476.

In recent years it has become evident that for most purposes in homotopy theory it is more convenient to use semi-simplicial complexes instead of topological spaces. As the work of the author and others has shown, a semi-simplicial complex satisfying the extension condition has all of the important properties of the total singular complex of a topological space. [Relevant material may be found in D. M. Kan, Proc. Nat. Acad. Sci. U.S.A. **42** (1956), 419–421; MR **19**, 440; J. W. Milnor, Ann. of Math. (2) **65** (1957), 357–362; MR **18**, 815; and J. C. Moore, Semi-simplicial complexes and Postnikov-systems, Proceedings of the International Symposium on Algebraic Topology and its Applications (1956), Mexico.]

A semi-simplicial complex is said to satisfy the extension condition if, given n simplexes of dimension $(n-1)$ of the complex which satisfy conditions so that they appear to be all of the boundary save for a single face of an n-simplex, there is an n-simplex in the complex for which they are precisely all of the boundary save for a single face. For such complexes, usually called Kan complexes, homotopy groups may be defined, and, further, the notion of homotopy between maps of an arbitrary semi-simplicial complex into a Kan complex is an equivalence relation.

In this paper the author proves that any semi-simplicial complex may be canonically imbedded in a Kan complex with the same homology groups, and that this process is functorial on the category of semi-simplicial complexes and maps. Further, if the original complex is a Kan complex, then it has the same homotopy type as the new one.

For any semi-simplicial complex K, let $\mathrm{Ex}^\infty K$ denote the Kan complex associated with K in the fashion described by the author, and if $f: K \to L$ let $\mathrm{Ex}^\infty f: \mathrm{Ex}^\infty K \to \mathrm{Ex}^\infty L$ denote the map induced by f. If $f: K \to L$ in a fibre map with fibre F, then it is proved that $\mathrm{Ex}^\infty f: \mathrm{Ex}^\infty K \to \mathrm{Ex}^\infty L$ is a fibre map with fibre $\mathrm{Ex}^\infty F$. This result and others, such as the fact that the singular complex of the geometric realization of a semi-simplicial complex K and $\mathrm{Ex}^\infty K$ have the same homotopy type, indicate the usefulness of the functor Ex^∞. *J. C. Moore.*

Inoue, Yoshiro. A note on the singular homotopy type of spaces. Proc. Japan Acad. **33** (1957), 90–91.

In this paper the author proves that if X and Y are p-connected spaces, then if $\Omega(X)$ and $\Omega(Y)$, the loop spaces in X and Y, have the same singular q-homotopy type for some integer q such that $p \leq q \leq 2p-1$, then X and

Y have the same singular $(q+1)$-homotopy type. This result is well known to those familiar with Postnikov systems and Postnikov invariants, and is an immediate corollary of the fact that in the stated range of dimension the Postnikov invariants are stable under the cohomology suspension homomorphism. *J. C. Moore.*

Inoue, Yoshiro. A note on homotopy classification and extension. Proc. Japan Acad. **33** (1957), 87–89.

Let Y be a space such that $\pi_i(Y)=0$ for $0\leqq i<n$, $n<i<q$, $q<i<r$, where $n>1$. Mizuno [J. Inst. Polytech. Osaka City Univ. Ser. A. **6** (1955), 55–69 (we refer to this paper as M); MR **17**, 994] has obtained expressions for the third obstruction to extensions and homotopies of maps of a complex K into Y when $r<2q-1$. In this note the author solves the same problems when $r\geqq 2q-1$; the notations are those of M. The resulting formulae closely resemble those of M; in fact, the difference lies in a modification of the y_γ-operation of M. The latter depended on a chain transformation $\gamma_{n,q}$: $K,\ L\to K(\pi,\ n,\ \pi',\ q,\ k)$, of degree $(n-n_1)+(q-q_1)$, which itself depended on the pair $x_{n_1}\in Z^{n_1}(K,\ L_1;\ \pi)$, $x_{q_1}\in Z^{q_1}(K,\ L_2;\pi')$. This the author replaces by a chain transformation $R_{n,q}$: $K,\ L\to K(\pi,\ n,\ \pi',\ q,\ k)$ of degree $q-q_1$, which depends on a pair $x_n\in Z^n(K,\ L_0;\ \pi)$, $c_q\in C^q(K,\ L_0;\ \pi')$ satisfying condition (1.2) of M, and on $x_{q_1}\in Z^{q_1}(K,\ L_1;\ \pi')$. (Note: $L=L_1\cup L_2$ in M, $L=L_0\cup L_1$ in this note, c_q appeared as x_q in M.) The procedures for defining $\gamma_{n,q}$ and $R_{n,q}$ are essentially similar. Then an operation $y_n(x_n,\ c_q;\ x_{q_1})$ is defined analogously to y_γ in M and Lemma 3, Theorem 1 and Theorem 2 of this note are then the expected modifications of Theorems 7.2, 7.4 and 8.1 of M. {Reviewer's comment: The author defines a more general transformation $R_{n,q}$ depending on x_n, c_q and $x_{q_1},\ \cdots,\ x_{q_r}$. No application is made of this generality, which serves only to mask the resemblance of $R_{n,q}$ to $\gamma_{n,q}$. Moreover, the number r occurring here is not related to the number r which appears in the hypothesis on Y.} *P. J. Hilton.*

Dugundji, J. Continuous mappings into nonsimple spaces. Trans. Amer. Math. Soc. **86** (1957), 256–268.

The paper deals with the homotopy classification of maps of a space X into a non-simple space Y. This problem was fully investigated by Olum [Ann. of Math. (2) **52** (1950), 1–50; MR **12**, 120], using obstruction theory with local coefficients. If h: $\pi_1(X)\to\pi_1(Y)$ is geometrically realizable, it is easy to see that the homotopy classes of maps $X\to Y$ inducing h are in one-one correspondence with equivariant homotopy classes of equivariant maps $\tilde{X}\to\tilde{Y}$, where $\tilde{X}$, $\tilde{Y}$ are appropriate covering spaces. Using this remark, the author transposes some of Olum's results, which he states in terms of equivariant cohomology groups of $\tilde{X}$ in place of cohomology groups of X with local coefficients. *J.-P. Meyer.*

Kervaire, Michel A. Relative characteristic classes. Amer. J. Math. **79** (1957), 517–558.

The author considers an S_{n-1}-sphere-bundle over a simplicial complex K with $O(n)$ as structural group, and assumes that over a subcomplex L a cross-section θ^r is given in the associated bundle with fibre the Stiefel manifold $V_{n,n-r}$ (of $(n-r)$-frames in euclidean n-space); for $q\geqq r$ he then defines (by the usual stepwise extension process of the cross-section) relative Stiefel-Whitney classes, which are elements of $H^{q+1}(K,\ L;\ \mathbf{Z}_2)$. Extending to the relative classes the definition of characteristic classes as symmetric functions due to A. Borel [Comment. Math. Helv. **27** (1953), 165–197; MR **15**, 244], the author proves that a duality theorem (similar to the Whitney duality theorem for the usual classes) holds for the relative classes. He introduces Wu classes U^p [satisfying $\mathrm{Sq}^p(X_R^{n-p})=U^p\cdot X_R^{n-p}$ for all $X_R^{n-p}\in H^{n-p}(K,\ L;\ \mathbf{Z}_2)$] by duplicating K, with the two copies of L pointwise identified, and extends Wu's theorem [C. R. Acad. Sci. Paris **230** (1950), 508–511; MR **12**, 42] (expressing the Stiefel-Whitney classes of the tangent bundle by means of Steenrod squares of the Wu classes) to the case of manifolds K with a regular boundary L. The same formula also expresses relative Stiefel-Whitney classes in terms of relative Wu classes (defined when a cross section over a subcomplex L is given in the associated principal bundle).

Let M_d be a differentiable closed manifold imbedded into euclidean space E_{d+n}, and assume the existence of a continuous field $\mathbf{F}_n$ of n-frames normal to M_d. Using the mapping of the tubular neighborhood (in E_{d+n}) of M_d into the normal planes (provided with the frames of $\mathbf{F}_n$), an element of $\pi_{d+n}(S_n)$ is attached to the imbedding of M_d into E_{d+n}. Attaching a map ω: $M_d\to V_{d+n,n}$ (sending the frames at each point of M_d into the Stiefel manifold $V_{d+n,n}$ at the origin in E_{d+n}), the generalized curvature integra c describes the class of $\omega(M_d)$ in $H_d(V_{d+n,n};\ \mathbf{Z})$ (the latter group being infinite cyclic or cyclic of order 2). Defining $\chi^*(M_d)$ as being one half the characteristic of M_d if d is even, and $\sum_{i=0}^{k}(-1)^i p_i$ for d odd [with p_i=rank of $H_i(M_d;\ \mathbf{Z}_2)$ and k=largest integer in $d/2$], and defining $\gamma=c-\chi^*(M_d)$, the author then proves that γ equals the generalized Hopf invariant [in the sense of Steenrod, Ann. of Math. (2) **50** (1949), 954–988; MR **11**, 122] of the element of $\pi_{d+n}(S_n)$ corresponding to the imbedding. The proof is given for $n=d+1$, as it can be extended to other dimensions by suspension maps; it uses the construction of an n-manifold with M_d as its boundary and the computation of the relative Stiefel-Whitney classes for a field of frames which coincides with $\mathbf{F}_n$ on M_d. As a consequence, if $n\leqq d$, $c=\chi^*(M_d)$ and does not defend on the imbedding nor on the n-field. If $\pi_{2d+1}(S_{d+1})$ contains no map with odd Hopf invariant (which always happens if $d\neq 2^a-1$), then $c=\chi^*(M_d)$.

The author also gives an axiomatic description of relative Chern classes [as elements of $H^{2(q+1)}(X,\ A;\ \mathbf{Z})$, starting from the absolute Chern classes (and the classifying bundle), whereas the relative Stiefel-Whitney classes were defined using obstructions to the extension of cross-sections. Both methods would apply in both cases, as well as the construction by means of symmetric functions. A Whitney duality theorem holds for relative Chern classes. The author defines relative Pontrjagin classes (with coefficients in $\mathbf{Z}_p$, p prime >2) which also satisfy a Whitney duality property and appear as symmetric functions. *G. Hirsch* (Brussels).

Sanderson, D. E. Isotopy in 3-manifolds. I. Isotopic deformations of 2-cells and 3-cells. Proc. Amer. Math. Soc. **8** (1957), 912–922.

This paper, the first of a series, is concerned primarily with the existence of (1) certain "simplicial isotopies" on triangulated 3-manifolds and (2) "shellable" subdivisions of certain 3-cells on triangulated 3-manifolds with boundary.

If K and M are two triangulated spaces, a piecewise linear isotopy is a mapping $G(x,\ t)$ of $K\times[0,\ 1]$ into M which, for each $t\in[0,\ 1]$ is a piecewise linear homeomorphism between two subdivisions K' and M' of K and M. $G(x,\ t)$ is simplicial (with respect to K') if it is piece-

wise linear relative to K' for each t. The isotopy theorems affirm the existence of simplicial isotopies carrying polyhedral 2-cells and 3-cells on triangulated 3-manifolds into individual simplexes of their respective subdivisions. The isotopies are fixed outside polyhedral neighborhoods of the points which necessarily move. If K is an n-manifold with boundary and $S \subset K$, then S can be shelled from K if there exists a cellular subdivision of S whose n-cells in some order $(C_1, C_2, \cdots)$ can be removed one at a time, where C_t always intersects F in an $(n-1)$-cell on the boundary of the closure of $K - \mathbf{U}_{j<t} C_j$. The shelling theorems affirm the existence of shellable subdivisions of suitably restricted polyhedral 3-cells on triangulated 3-manifolds with boundary. *S. S. Cairns.*

Papakyriakopoulos, C. D. On Dehn's lemma and the asphericity of knots. Ann. of Math. (2) **66** (1957), 1–26.

Detailed proofs of the previously announced results [Proc. Nat. Acad. Sci. U.S.A. **43** (1957), 169–172; MR **18**, 590]. These include Dehn's lemma, the asphericity of knots, Whitehead's conjecture, Higman's problem, Hopf's conjecture, and a new, important, result that the author calls the Sphere Theorem. The statements of these theorems may be found in the review quoted above. There is a common theme, which is perhaps best exemplified in the Sphere Theorem, a kind of "absolute" Dehn Lemma. Although the demonstrations are reasonably simple and very elegant, they have already been simplified and extended (by A. Shapiro, J. Milnor and J. H. C. Whitehead; in part in process of publication).

Although it is not yet possible to evaluate the impact of this important paper on the development of geometric topology, it has already led to renewed attack on the problem of classifying the 3-dimensional manifolds; significant results have been and are being obtained. A complete solution has suddenly become a definite possibility.

Solution of these problems, especially of the Dehn Lemma problem, fills a gap of long standing in our knowledge of 3-dimensional spaces. However, the importance of this paper lies even more in the method used. The geometry of covering spaces is exploited heavily, underscoring again the basic nature of this construction. This is a paper which must be read by every student of 3-dimensional topology. *R. H. Fox* (Princeton, N.J.).

Kinoshita, Shin'ichi. On Wendt's theorem of knots. Osaka Math. J. **9** (1957), 61–66.

Let $M_g(k)$ be the g-fold branched cyclic covering of a tame knot k, and let e_g be the rank of the smallest free abelian group that can be mapped homomorphically upon $H_1(M_g(k))$. Suppose that in a knot k a 2-string braid that is an even power of the generator σ of the braid group B_2 can be isolated. Denote by k' the knot obtained by replacing this braid by the trivial 2-string braid σ^0. By the weak unknotting number $\bar{s} = \bar{s}(k)$ of k is meant the length of the shortest sequence $k', k'', k''', \cdots, k''^{\cdots'}$ for which $k''^{\cdots'}$ is of trivial type. (It is obvious that $\bar{s} < \infty$). The author proves that $e_g \leqq \bar{s}(g-1)$. This generalizes a theorem due to Wendt [Math. Z. **42** (1937), 680–696]. *R. H. Fox* (Princeton, N.J.).

Švarc, A. S. Some estimates of the genus of a topological space in the sense of Krasnosel'skiĭ. Uspehi Mat. Nauk (N.S.) **12** (1957), no. 4(76), 209–214. (Russian)

A topological space X, on which a finite group G acts without fixed points, is said to be of kind n with respect to the action of G if n is the least positive integer for which there can be found n closed sets $A_1, \cdots, A_n$ such that (a) A_i does not intersect $g(A_i)$ unless $g=1$, and (b) $X = \mathbf{U}_{i=1,\cdots,n; g \in G} g(A_i)$. This invariant of X, G had already been considered for G cyclic of order 2 [Yang, Ann. of Math. (2) **60** (1954), 262–282; **62** (1955), 271–283; MR **16**, 502; **17**, 289] and for G finite cyclic [Krasnosel'skiĭ, Mat. Sb. N.S. **37(79)** (1955), 301–322; MR **18**, 139]. If X (which is always a regular covering of the collapsed space X') is the universal covering of X', then the kind of (X, G) reduces to the 1-dimensional category $h_1\text{cat}\ X'$ of X' [cf. Fox, Ann. of Math. **42** (1941), 333–370; MR **2**, 320]. Denote by p^* the homomorphism of the cohomology rings $H^*(X')$ into $H^*(X)$ induced by the projection $p: X \to X'$. The main result appears to be Th. 4: If in the kernel of p^* there can be found n elements $\xi_1, \cdots, \xi_n$ such that $\xi_1 \cdots \xi_n \neq 0$, then $\text{kind}(X, G) \geqq n+1$. If X is the universal covering of X', this reduces to an interesting dimensionalization of a well-known theorem of Schnirelmann [Fox, ibid. § 28]. *R. H. Fox* (Princeton, N.J.).

Milnor, John. Groups which act on S^n without fixed points. Amer. J. Math. **79** (1957), 623–630.

The main concern of the paper is the theorem, for the mod 2 homology sphere $M = M^n$, that if T is a fixed-point free (fpf) involution and f is an odd degree map of $M \to M$, then for some x, $Tfx = fTx$. The idea of the proof is that otherwise f could be factored through M/T, whence f_* would be trivial. An application is: In a finite group G of fpf transformations of M, the elements of order 2 are in the center (whence certain groups are eliminated as possible fpf transformation groups, or, for M^3, as possible fundamental groups). The paper calls attention to the problem: Can an abelian group of order 21 be fpf on a sphere—an answer in the negative would show that a result of Zassenhaus characterising fpf groups acting orthogonally on spheres is invalid if the orthogonality restriction is dropped. *D. G. Bourgin* (Paris).

See also: Banach Spaces, Banach Algebras, Hilbert Spaces: Borisovič. **Calculus of Variations:** Al'ber. **General Topology:** Hayashi.

GEOMETRY

Geometries, Euclidean and Other

Thébault, Victor. A propos du tranchet d'Archimède. Enseignement Math. (2) **3** (1957), 141–149.

★ **Pedoe, D. Circles.** International Series of Monographs on Pure and Applied Mathematics, Vol. 2. Pergamon Press, New York-London-Paris, 1957. x+78 pp. $3.75.

The first chapter of this attractive little book contains three proofs of the existence of the nine-point circle and several applications of inversion, including a neat proof of Feuerbach's Theorem, Gergonne's construction for a circle touching three given circles, and a new treatment of the geometry of compasses. In the second chapter the circle $x^2+y^2+2gx+2fy+c=0$ is represented by the point (g, f, c). From the theorem that four skew lines having more than two transversals have infinitely many, is deduced a corresponding statement about the number of solutions of the following problem. Given four pairs of

points P, P', Q, Q', R, R', S, S' in a plane, find another pair X, X' such that all the quadrangles $PP'XX'$, $QQ'XX'$, $RR'XX'$, $SS'XX'$ are cyclic.

The third chapter deals with the Argand diagram, groups of Möbius transformations, and Poincaré's conformal model of the hyperbolic plane. In the final chapter, Steiner's proof of the isoperimetric property of the circle is amplified by a careful discussion of the precise meaning of perimeter and area. *H. S. M. Coxeter.*

★ **Vivier, Marcel. Cours de calcul vectoriel. Tome I. Le calcul vectoriel en connexion avec quelques développements sur l'algèbre linéaire et les transformations. Préface de G. Bouligand.** 2e édition augmentée d'une note sur la réduction des matrices. Société d'Edition d'Enseignement Supérieur, Paris, 1955. i+651+xii pp. (polycopiées)

A text for students of the "Ecoles Techniques". The geometric point of view is stressed, with gradual passage to the algebraic. There are 10 chapters, on the topics: Vecteurs libres, points cotés (ou pondérés), vecteurs glissants, extension algébrique des méthodes vectorielles, notions sur les transformations, transformations corrélatives (formes bilinéaires et quadratiques), transformations métriques, compléments sur l'homographie (application aux coniques), l'homographie sur les quadriques et dans les faisceaux, compléments à la théorie des quadriques (groupes de transformations du plan, géométrie non euclidienne).

Janekoski, V. Sur les deux identités vectorielles. Bull. Soc. Math. Phys. Macédoine 7 (1956), 51–53. (Serbo-Croatian. French summary)

Concise demonstrations are given of the two vector identities

$$(a\times b)\times c+(b\times c)\times a+(c\times a)\times b$$
$$=(a+b)\times c+(b+c)\times a+(c+a)\times b=0,$$

which have applications in geometry; see, e.g., Chattelun, L., Calcul vectoriel, Tome I, p. 216 [Gauthier-Villars, Paris, 1952; MR **14**, 361].

Ruse, H. S. On the geometry of $\mathscr{E}$-matrices. Proc. Roy. Soc. Edinburgh. Sect. A. **64** (1954), 127-144.

Es seien e_i die Einheiten der Quaternionen. Dann ist (1) $e_ie_j=\gamma_{ij}{}^ke_k$. Die Matrizen $\mathscr{E}_{ij}=R_iS_j$, wo $R_i=\gamma_{i0}{}^k$ und $S_j=\gamma_i{}^{k0}$ ist, bilden ein bekanntes System von 16 Matrizen, das von Milne angewandt worden ist für seine physikalische Theorie. Es gilt (2) $\mathscr{E}_{ijkl}=\mathscr{E}_{klij}$. Die Geometrie dieses Systems wird verdeutlicht, indem Verf. das System auf eine ganz andere Weise ableitet. Die linearen Transformationen in P_3, die eines der zwei Geradensysteme einer Quadrik $g_{ij}x^ix^j=0$ invariant lassen, können in der Form $Y=rX$, $r^j{}_k=\alpha^i\gamma_i{}^j{}_k$ geschrieben werden, wo γ von g_{ij} und von den Koordinaten eines Basispunktes C^i abhängig sind. Für $g_{ij}=\delta_{ij}$ und C^i (1, 0, 0, 0) erhält man die γ der Gleichung (1). Auf diese Weise bekommt man verallgemeinerte $\mathscr{E}$-Matrizen, für die auch (2) gilt.

J. Haantjes (Zbl **57** (1956), 366).

★ **Grotemeyer, Karl Peter. Analytische Geometrie.** Sammlung Göschen Bd. 65/65a. Walter de Gruyter & Co., Berlin, 1958. 202 pp. DM 4.80.

There are 12 sections on the following topics: historical introduction, coordinate system, lines and planes, spheres, matrix calculus, affine transformations, motions, similarity transformations, surfaces of second order, projective geometry of space, projective quadrics.

★ **Tresse, A. Théorie élémentaire des géométries non euclidiennes.** Tome I. Gauthier-Villars, Paris, 1957. 151 pp. 2500 francs.

This volume is the first of two projected volumes of a text-book based upon the two articles of the author in Bull. Soc. Math. France **81** (1953), 81–143 and **83** (1955), 1–56 [MR **15**, 246; **17**, 655].

Kárteszi, Franz. Eine Bemerkung über das Dreiecksnetz der hyperbolischen Ebene. Publ. Math. Debrecen **5** (1957), 142–146.

Consider, in the hyperbolic plane, the regular tesselation $\{3, m\}$ $(m>6)$ consisting of equilateral triangles, m at each vertex. Choose any triangle and call it R_0. Denote by R_1 the ring of triangles surrounding R_0, i.e., those triangles (other than R_0) which have at least one point in common with R_0. Let R_i $(i>0)$ denote the ring of triangles surrounding R_{i-1}. Let z_i be the number of triangles in R_i. The author shows that $z_0=1$, $z_1=3(m-2)$, $z_2=3(m-2)(m-4)$, $z_i=(m-4)z_{i-1}-z_{i-2}$ $(i>2)$, and

$$\lim_{n\to\infty}\frac{z_n}{z_0+\cdots+z_n}=\frac{\sqrt{((m-4)^2-4)}-(m-6)}{2}$$

W. Moser (Saskatoon, Sask.).

Sasayama, Hiroyoshi. On the n-dimensional generalized quasi euclidean space. Comment. Math. Univ. St. Paul. **6** (1957), 49–70.

The paper is a sequel to a previous work [same Comment. **5** (1956), 95–114; MR **18**, 816]. The author develops some more topics of the analytic geometry of the quasi-euclidean n-dimensional space (linear subspaces, hyperspheres and volume of parallelotopes). *L. A. Santaló.*

★ **Чувиков, Н. Т. [Čuvikov, N. T.] Преобразование ортогональ'ных проекций. [Transformation of orthogonal projections.]** Gosudarstv. Izdat. "Sovetskaya Nauka", Moscow, 1957. 176 pp. 4.30 rubles.

Applications to descriptive geometry.

★ **Haack, Wolfgang. Darstellende Geometrie. I. Die wichtigsten Darstellungsmethoden. Grund- und Aufriss ebenflächiger Körper.** Zweite, durchgesehene und ergänzte Aufl. Sammlung Göschen Bd. 142. Walter de Gruyter & Co., Berlin, 1958. 113 pp. DM 2.40.

The first edition was reviewed in MR **15**, 980.

Convex Domains, Integral Geometry

Kuiper, N. H. On convex sets and lines in the plane. Nederl. Akad. Wetensch. Proc. Ser. A. **60**=Indag. Math. **19** (1957), 272–283.

The author's results concern existence of common secants for plane convex sets, a topic treated earlier by Santaló, Vincensini, Klee, Grünbaum, Hadwiger, and others [see esp. Grünbaum, Pacific J. Math. **5** (1955), 351–359; MR **17**, 185]. A closed subset G of the real projective plane P is called strongly convex provided G's intersection with any line in P is empty, a point or a closed interval. For a family F of strongly convex subsets of P, two properties are considered, as follows: B_i — for each i members of F there is a common secant (i.e., a line meeting all i of them); C_j — for any j members of F, no cycle homologous to a line in P can be formed from the lines which meet at least two of these j sets. The principal results assert that B_n (for arbitrary n) is implied by the

conjunction of B_3 and C_4, and also by the conjunction of B_4 and C_3. An example shows that conjunction of B_3 and C_3 is insufficient. The proofs utilize the relationship between strong convexity in P and ordinary convexity in the affine planes of P, a matter studied earlier by D. Dekker [Amer. Math. Monthly **62** (1955), 430–431; MR **16**, 950] and others. Also considered is the relationship of C_j to various other properties, including one introduced by Grünbaum (loc. cit.). The author's results yield corollaries for the affine plane, and these are sharper than earlier results. *V. L. Klee* (Seattle, Wash.).

Florian, August. Ungleichungen über Sternpolyeder. Rend. Sem. Mat. Univ. Padova **27** (1957), 16–26.

An inequality proved by the author for convex polyhedra inscribed in the unit sphere [Monatsh. Math. **60** (1956), 130–156; MR **17**, 1235] is generalized to a certain class of starpolyhedra (not to be confused with star-shaped polyhedra) inscribed in the unit sphere.

The regular starpolyhedra are the only ones in this class for which the inequality becomes an equality. This is analogous to the result of Fejes Tóth [Acta Math. Acad. Sci. Hungar. **7** (1956), 31–48; MR **18**, 63] where the regular starpolyhedra are characterized by a similar inequality for starpolyhedra circumscribed to the unit sphere. [See the latter paper for definitions and illustrations.] *E. G. Straus* (Los Angeles, Calif.).

Fejes Tóth, L. Filling a domain by isoperimetric discs. Publ. Math. Debrecen **5** (1957), 119–127.

A remark of D'Arcy W. Thompson [On growth and form, vol. 2, 2nd ed., Cambridge, 1952, p. 471] suggests the problem of finding an upper bound for the total area T of n isoperimetric convex discs, each of perimeter λ, which can be placed in a given convex region of area H without overlapping. Using the abbreviation

$$\rho=(\lambda/2\pi)\sqrt{(n/H)},$$

the author remarks that, when

$$\rho<2^{-\frac{1}{2}}3^{-\frac{1}{4}}=0.537\cdots,$$

the discs can be circular, so that T/H has the obvious upper bound $\pi\rho^2$; and when

$$\rho=2^{\frac{1}{2}}3^{\frac{1}{4}}\pi^{-1}=0.592\cdots,$$

the discs can be hexagons forming a regular tessellation, so that $T/H=1$. The problem is significant for intermediate values of ρ, where it is found that T/H has the upper bound

$$(2^{3/2}3^{\frac{1}{2}}\rho-\pi\rho^2-\pi)\pi/(2\cdot3^{\frac{1}{2}}-\pi).$$

H. S. M. Coxeter (Toronto, Ont.).

Helfenstein, H. G. An extremal problem concerning the centre of gravity of a convex disc. Proc. Amer. Math. Soc. **8** (1957), 992–1001.

The following problem is solved: Find, among all convex disks in E^2 with diameter $\leqq d$, one for which the center of gravity has maximal distance from a chord of length a, $0<a\leqq d$. A pentagon $\pi=ABCDE$ of the following type provides a solution. AB, BC, EA are straight segments and AB has length a. Reflection in the perpendicular bisector of AB takes π into itself. The sides CD and DE of π are arcs of circles with radius d about A and B respectively. The angle $2\vartheta=\angle DAC=\angle DBE$, which determines π completely, is evaluated in terms of $2\alpha=\angle ADB$. Corollaries are: The distance of any chord of a convex disk R of diameter d from the center of gravity G of R is at most $2d/3$. Any diameter of R has distance at most γd from G, where $\gamma=0.3426\cdots$. Both bounds are sharp. *H. Busemann* (Los Angeles, Calif.).

Grünbaum, B. A simple proof of Borsuk's conjecture in three dimensions. Proc. Cambridge Philos. Soc. **53** (1957), 776–778.

It is proved that every set of diameter 1 in 3-space is the union of 4 sets each of diameter less than 1. A complicated proof of this fact was given by Eggleston [J. London Math. Soc. **30** (1955), 11–24; MR **16**, 734]. The present author's simple proof starts from the fact proved by the reviewer [Proc. Amer. Math. Soc. **4** (1953), 222–225; MR **14**, 787] that any set of diameter 1 can be covered by a regular octahedron of diameter $\sqrt{3}$. He then chops off three of the corners of the octahedron and shows that the resulting polyhedron still has the universal covering property and can be dissected into 4 sets each of diameter.

$$\frac{(6{,}129{,}030-937{,}419\sqrt{3})^{\frac{1}{2}}}{1518\sqrt{2}}=.9887.$$

D. Gale (Santa Monica, Calif.).

Steinhaus, H. Sur la division des corps matériels en parties. Bull. Acad. Polon. Sci. Cl. III. **4** (1956), 801–804 (1957).

Le problème principal de cette Note est la division d'un corps Q en n parties K_j ($j=1, 2, \cdots, n$) et le choix de n points A_j de manière à rendre aussi petite que possible la somme $S(K, A)=\sum_{j=1}^{n} I(K_j, A_j)$ ($K\equiv\{K_j\}$, $A\equiv\{A_j\}$), où $I(Q, P)$ désigne, en général, le moment d'inertie d'un corps quelconque Q par rapport à un point quelconque. D'après les résultats de cette Note, la solution de ce problème, Q et n étant donnés, est en général unique, mais il y a des corps avec une infinité de solutions. Encore, l'auteur discute quelques applications pratiques de ce problème. *M. Kondô* (Tokyo).

See also: Functions of Complex Variables: Falgas. **Geometries, Euclidean and Others:** Pedoe.

Differential Geometry

★ **Norden, A. P. Differentialgeometrie. II.** VEB Deutscher Verlag der Wissenschaften, Berlin, 1957. viii+124 pp. DM 7.40.

A translation by Rolf Sulanke of the original Russian book published by Gosudarstv. Uč.-Ped. Izdat., Moscow, 1948. Part I was listed in MR **19**, 58. In this translation of the second part the solutions to the exercises are omitted.

Santaló, L. A. On the uniqueness of elementary vector operators. Math. Notae **14** (1955), 120–132 (1956). (Spanish)

The author characterises, up to constant factors, the 3-dimensional vector operators grad, div and curl. Let the rectangular cartesian coordinates of a point be x, and let O be an orthogonal matrix. A scalar (scalar density) is a function $F(x)$ such that $F(Ox)=F(x)$ $(=\det O\cdot F(x))$; a set A of three functions is a vector (vector density) if $A(Ox)=OA(x)$ $(=\det O\cdot OA(x))$. The author proves that the only vectors whose components are linear combinations, with constant coefficients, of the first partial derivatives of a scalar are proportional to the gradient. There

are no (non-zero) vector densities of this nature. Similar theorems are given for div and curl. The proofs are elementary and matrix-theoretic, the basic result being a particular case of Schur's lemma. *J. H. Williamson.*

Calapso, Renato. Sul teorema di permutabilità di Bianchi per le trasformazioni asintotiche delle superficie. Rend. Sem. Mat. Messina **1** (1955), 146–148.

L'A. mette in relazione i determinanti ortogonali in sei variabili di Guichard [Théorie générale des reseaux, Mémor. Sci. Math. no. 77, Gauthier-Villars, Paris, 1936], con il teorema di permutabilitá di Bianchi per le trasformazioni asintotiche delle superficie. — Si enuncia, senza dimostrazione, che il metodo seguito dall'A. dá la piú generale configurazione di Bianchi relativa al problema in oggetto. *V. Dalla Volta* (Roma).

Dorfman, A. G. Solution of the bending equation for certain classes of surfaces. Uspehi Mat. Nauk (N.S.) **12** (1957), no. 2 (74), 147–150. (Russian)

An analytic surface $z=f^{(n)}(x, y)+f^{(n+1)}(x, y)+\cdots$, where $f^{(k)}(x, y)$ is a form of degree k in x, y, can for $n\geqq 5$ in general not be bent within the class of analytic surfaces [see N. V. Efimov, Trudy Mat. Inst. Steklov **30** (1949); MR **12**, 531]. The bendable surfaces form a set of measure 0. This paper gives some specific information on bendability. Every surface z for which one of the second order partial derivatives vanishes identically can be bent. For given m there is a surface z which can be bent under preservation of its osculating paraboloid $z=f^{(n)}(x, y)+\cdots+f^{(n+m)}(x, y)$. *H. Busemann* (Los Angeles, Calif.).

Vincensini, Paul. Sur les systèmes de ∞^2 sphères centrées sur une développable isotrope. J. Math. Pures Appl. (9) **36** (1957), 155–169.

For a congruence of spheres in imaginary Euclidean space of three dimensions, the angular element $d\phi$ is the angle between two consecutive spheres of the congruence. It is given by the differential quadratic form $d\phi^2=(ds^2-dR^2)/R^2$, where ds denotes the linear element on the central locus S and $R\neq 0$ is the radius of the congruence. In 1935, A. Demoulin defined the curvature K of such a congruence of spheres to be that of this differential quadratic form [Acad. Roy. Belg. Bull. Cl. Sci. (5) **21** (1935), 770–772]. In a previous article [same J. **16** (1937), 315–328], the author obtained a formula for K provided that the central locus S is neither an isotropic curve nor an isotropic plane nor an isotropic developable. These singular cases are studied in the present article. Thus the author obtains the complete solution of the problem of finding all congruences of spheres which preserve their curvature K when the radii R of the ∞^2 spheres are multiplied by an arbitrary constant $C\neq 0$.

J. De Cicco (Chicago, Ill.).

★ **Reichardt, Hans. Vorlesungen über Vektor- und Tensorrechnung.** Hochschulbücher für Mathematik, Bd. 34. VEB Deutscher Verlag der Wissenschaften, Berlin, 1957. xi+499 pp. DM 41.20.

The books has 12 chapters and an appendix. Its topics are: intuitive vector calculus, vector analysis, vector spaces of finite dimension over skew fields, vector spaces over commutative fields, metric of vector spaces, linear point-spaces, tensor algebra, metric of tensor spaces, tensor analysis, differentiable manifolds, spaces with affine connection, Riemann spaces, integral equations.

★ **Chern, Shiing-shen. Topics in differential geometry.** The Institute for Advanced Study, Princeton, N. J., 1951. 106 pp. (mimeographed) $3.00.

Segre, Beniamino. Sul differenziale delle forme esterne di classe zero. Rend. Sem. Mat. Messina **1** (1955), 1–9.

Sia ω_r una forma esterna di classe zero in n variabili; se esiste una forma σ_{r+1}, pure di classe zero, tale che per ogni campo E_{r+1} di E^n (euclideo) sia $\int_{aE_{r+1}}\omega_r=\int_{E_{r+1}}\sigma_{r+1}$, si dirà che σ_{r+1} è il differenziale esterno generalizzato di ω_r. — L'A. dimostra allora: (1) Sia $\{\overset{(j)}{\omega}_r\}$ $(j=1, 2, \cdots)$ una successione di forme di classe 1, definite in un medesimo n-campo di E^n, e tendenti ivi uniformemente ad ω_r; se, e soltanto se, la successione $\{\overset{(j)}{d\omega}_r\}$ ammette limite per $m\to\infty$, e se questo limite è raggiunto uniformemente, allora ω_r ammette differenziale generalizzato, e questo è dato dal limite sopra nominato. (2) Per il differenziale generalizzato valgono le ordinarie regole del calcolo esterno.

La dimostrazione data in questo lavoro riproduce, con qualche completamente e semplificazione, quella giá data dall'A. nella sua opera "Forme differenziali esterne e loro integrali" [v. 1, Edizioni Univ. Docet, Roma, 1951, pp. 186 e segg.; MR **14**, 208]. *V. Dalla Volta.*

Liber, A. E. On the theory of surfaces in a geometrical n-space with given fundamental group. Trudy Sem. Vektor. Tenzor. Anal. **10** (1956), 193–226. (Russian)

A group G_r of Lie, given by the transformations $'x^\lambda=f^\lambda(x^\mu, a^i)$ $(\lambda, \mu=1, 2, \cdots, n; i, j=1, \cdots, r)$ is given as the fundamental group of a space (x^λ); such a space is denoted by K_n. In the K_n an m-surface is given by $x^\lambda=x^\lambda(\eta^a)$ $(a=1, 2, \cdots, m)$. A theory of differential invariants of the G_r is developed; on the basis of this theory the fundamental theorems for the determination of the m-surface in the K_n are derived as relations between geometrical objects. This leads to equations which can be considered as generalizations of the Gauss-Codazzi equations (or, as they are called in the Russian literature, equations of Gauss-Peterson-Codazzi). The final section studies the bending of these m-spaces in the K_n. The theory is based on the theory of the differential object due to V. V. Vagner, explained in his preface to the Russian translation of the book by O. Veblen and J. H. C. Whitehead, "The foundations of differential geometry" [Moscow, 1949]. *D. J. Struik* (Cambridge, Mass.).

MacDuffee, C. C. Curves in Minkowski space. Duke Math. J. **24** (1957), 379–389.

The method of invariants is used to study the differential properties of curves in flat 4-space with signature $(+---)$. *A. G. Walker* (Liverpool).

See also: Harmonic Functions, Convex Functions: Reade. **Algebraic Topology:** Kervaire. **Manifolds, Connections:** Laugwitz.

Manifolds, Connections

Vranceanu, Gheorghe. Trasformazioni puntuali tra spazi affini o proiettivi e spazi a connessione affine euclidea. Boll. Un. Mat. Ital. (3) **12** (1957), 145–153.

On peut associer à un espace $A_n(x^1, \cdots, x^n)$ localement euclidien avec la connexion Γ_{jk}^i $(T_{jk}^i \overset{\text{def}}{=} \Gamma_{jk}^i-\Gamma_{kj}^i=0, \Gamma_{jki} \overset{\text{def}}{=} \partial\Gamma_{jk}^i/\partial x^k-\partial\Gamma_{ji}^i/\partial x^k+\Gamma_{sk}^i\Gamma_{ji}^s-\Gamma_{si}^i\Gamma_{jk}^s=0)$ une transformation ponctuelle

$$(*) \qquad u^i=u^i(x^1, \cdots, x^n),$$

où u^i sont les coordonnées cartésiennes de l'espace A_n.

En interprétant $A_n(x^1, \cdots, x^n)$ comme un espace affin euclidien et en considérant u^i comme les coordonnées d'un autre espace affin euclidien $E_n(u^1, \cdots, u^n)$, alors les équations (*) représentent une transformation ponctuelle entre les deux espaces A_n et E_n. Ensuite on peut considérer les coefficients Γ_{jk}^i comme les composantes d'un tenseur de l'espace A_n. Ainsi à chaque propriété de la connexion euclidienne Γ_{jk}^i corresponde une propriété de la transformation (*) et vice versa. Dans le travail actuel on étudie de ce point de vue les objects $\Gamma_k \overset{\text{def}}{=} \Gamma_{ik}^i$, $\Gamma_{kl} \overset{\text{def}}{=} \Gamma_{kp}^p \Gamma_{lo}^o$.

En interprétant $A_n(x^1, \cdots, x^n)$ et $E_n(u^1, \cdots, u^n)$ comme les espaces projectifs, les coefficients de la connexion projective de T. Y. Thomas, $\Pi_{jk}^i \overset{\text{def}}{=} \Gamma_{jk}^i - (n+1)^{-1}\delta_j^i \Gamma_k - (n+1)^{-1}\delta_k^i \Gamma_j$, représentent dans l'espace A_n les composantes d'un tenseur qui joue avec le tenseur associé $\Pi_{jk}^i \overset{\text{def}}{=} \Pi_{jl}^s \Pi_{ks}^l$ un rôle fondamental pour l'étude de la transformation (*).

En particulier, pour le cas $n=2$, l'auteur établit un théorème très intéressant: La transformation (*) (qui n'est pas une homographie) est de la première espèce, de la deuxième espèce ou de la troisième espèce (selon O. Borůvka) justement lorsque le tenseur Π_{jk} n'est pas dégénéré, est dégénéré (mais non identiquement nul) ou est identiquement nul. *A. Urban* (Prague).

Haimovici, M. Sur la représentation géométrique des systèmes mécaniques non holonomes. Schr. Forschungsinst. Math. **1** (1957), 280–285.

Un système holonome est représenté géométriquement par un espace de Riemann dont la métrique est $ds^2 = Tdt^2$, T étant l'énergie cinétique. Pour les systèmes non-holonomes, avec les liaisons cinématiques (B) $\omega_\sigma = a_{\sigma i}dx_i$ ($\sigma = m+1, \cdots, n$), Cartan [Atti Congresso Internaz. Mat., Bologna, 1928, t. IV, Zanichelli, Bologna, 1931, pp. 253–261] a été conduit aux espaces dont la métrique est définie seulement pour les déplacements (B) et dans lesquels est donnée en chaque point une normale à $n-m$ dimensions au plan à m dimensions défini par (B). Vrănceanu [Ann. Mat. Pura Appl. (4) **6** (1929), 9–43] a pourtant montré que ces espaces ne représentent pas toujours les systèmes non-holonomes. L'auteur analyse les conditions dans lesquelles cela a lieu et discute la représentation dans les autres cas. *O. Bottema* (Delft).

Soós, Gy. Über eine spezielle Klasse von Finslerschen Räumen. Publ. Math. Debrecen **5** (1957), 150–153.

The author generalizes the concept of symmetric spaces to Finsler geometry in terms of requiring that the covariant derivatives of certain tensors vanish. He is particularly interested in what he calls the quasi-symmetric case. For these spaces he proves that there always exists a positive dimensional continuous group of affine transformations. *L. Auslander* (Bloomington, Ind.).

Laugwitz, Detlef. Eine Beziehung zwischen affiner und Minkowskischer Differentialgeometrie. Publ. Math. Debrecen **5** (1957), 72–76.

In an n-dimensional centred affine space the author considers a closed hypersurface J. A function $F(x)$ is assumed defined everywhere, to be positive homogeneous and to satisfy certain differentiability conditions. The $F(x)$ serves to define a Minkowski metric on J by the condition $F=1$. The F gives rise to a fundamental metric tensor g_{ij} for the whole space, and hence a Riemannian induced metric $g_{\alpha\beta}$ is induced on J. On the other hand, there is defined on J a two-index metric tensor $s_{\alpha\beta}$ provided by affine surface theory.

The author proves that these two tensors only differ in sign. Some consequences for the relationship between affine and Finsler geometries are pointed out.

Finsler geometry has been approached from two points of view which the author's result serves to bring into contact with each other. One point of view, associated with the name of Cartan, regards Finsler geometry as locally Euclidean on the space of line elements. The other point of view, developed by Rund, Busemann and Barthel, regards Finsler geometry as locally Minkowskian. The fact that the Minkowski metric defined on an indicatrix J also leads to an induced Riemannian metric on J suggests that the Cartan theory can be interpreted in terms of the locally Minkowskian theory developed by the other authors. *E. T. Davies* (Southampton).

Moór, Arthur. Über die Torsions- und Krümmungsinvarianten der dreidimensionalen Finslerschen Räume. Math. Nachr. **16** (1957), 85–99.

In his treatment of two-dimensional Finsler spaces F_2, Berwald [Ann. of Math. (2) **42** (1941), 84–112; MR **2**, 304] achieved considerable simplifications by making use of a pair of unit orthogonal vectors defined for each line-element of F_2. In the present paper this method is extended, in as much as three mutually orthogonal vectors e_i, e_i, e_i are constructed with respect to each line-element $(x^i, \dot{x}^i)$ of a three-dimensional Finsler space F_3. The "principal scalar" (Hauptskalar) J of Berwald is generalised by means of the definition

$$A_{ijk} = \underset{(\alpha\beta\gamma)}{J}\, \underset{(\alpha)}{e_i}\, \underset{(\beta)}{e_j}\, \underset{(\delta)}{e_k},$$

where A_{ijk} is the torsion tensor. These quantities are used to describe the variation of the length of vectors undergoing parallel displacement (in the sense of Berwald); they are also useful for the characterisation of the so-called affinely connected spaces (these are spaces for which the connection parameters $\Gamma^*{}_j^i{}_k$ of Cartan are independent of direction). Landsberg spaces are defined to be Finsler spaces for which $A_{ijk|0}=0$ (the vertical bar denoting the covariant derivative of Cartan). It is shown that 3-dimensional Landsberg spaces of constant curvature are either Riemannian spaces or spaces of vanishing curvature. More generally, spaces F_3 of constant or of isotropic curvature are characterised in terms of the representation of the curvature tensor with respect to the orthogonal frames. *H. Rund* (Durban).

See also: Calculus of Variations: Al'ber. **Lie Groups and Algebras:** Horie. **Differential Geometry:** Reichardt; MacDuffee. **Complex Manifolds:** Martinelli.

Complex Manifolds

Martinelli, Enzo. Sulle varietà kähleriane dotate di isotropia caratteristica. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) **21** (1956), 400–404.

If the Riemann curvature tensor is constant at a point (only),

$$R_{hijk} = \tfrac{1}{2}K(g_{hk}g_{ij} - g_{hj}g_{ik}),$$

then for any surface element q^{jk},

$$-R_{hijk}q^{jk} = Kq_{hi};$$

so that, as known, the infinitesimal rotation of the co-

ordinate frame produced is parallel to q^{jk}, and its size is the (infinitesimal) area of q^{jk}.

For a Kaehler metric, if (in the terminology of this reviewer) the holomorphic curvature is constant at a point,

$$R_{\alpha\beta^*\gamma\delta^*}=\tfrac{1}{2}K(g_{\alpha\delta^*}g_{\gamma\beta^*}+g_{\alpha\beta^*}g_{\gamma\delta^*}),$$

then the author obtains the following. For a holomorphic surface element $q^{\gamma\delta^*}$ ("faccetta caratteristica") one has

$$-R_{\alpha\beta^*\gamma\delta^*}q^{\gamma\delta^*}=(K/2)q_{\alpha\beta^*}+i(K/2)\sigma g_{\alpha\beta^*},$$

and the interpretation is that the rotation produced is the sum of two addends. The first is again parallel to $q^{\gamma\delta^*}$, but of half-size $(K/2)\sigma$. The second is independent of the given surface element, and it displaces any holomorphic element into itself by an angle $(K/2)\sigma$. *S. Bochner.*

Rothstein, Wolfgang. Zur Theorie der analytischen Mannigfaltigkeiten im Raume von n komplexen Veränderlichen. Math. Ann. **133** (1957), 271–280.

In 1906 [Math. Ann. **62** (1906), 1–88] F. Hartogs discovered the phenomenon of simultaneous holomorphic continuation of holomorphic functions of several variables. There is the "Kontinuitätssatz"; and the "theorem of Hartogs-Osgood": Any function holomorphic on the boundary ∂G of a domain G can be continued into all of G if ∂G is connected. The author has shown in two recent papers [ibid. **122** (1951), 424–434; **129** (1955), 96–138; MR **12**, 818; **17**, 84] that similar continuation properties hold for analytic sets if G is "r-convex" (a notion introduced (by the author). (A set is an analytic set if it is locally the simultaneous zero variety of one or several holomorphic functions.)

In the present paper he sharpens (with respect to the dimension relation) the previously obtained results. Let G and G_0, $G_0 \subset\subset G$, be domains in the C^n. Let G be r-convex. Let $\mathfrak{g}^k$ be a set analytic and irreducible in $G-G_0$ and of complex dimension k. Let $k+r \geqq n+1$. Let $\mathfrak{g}^k$ get arbitrarily close to ∂G. Then $\mathfrak{g}^k$ can be continued to an analytic set $\mathfrak{g}_*{}^k$ in G such that there exists a domain $G^* \subset\subset G$ such that $\mathfrak{g}_*{}^k \equiv \mathfrak{g}^k$ in $G-G^*$. $\mathfrak{g}_*{}^k$ is unique (in a certain sense). It is further shown that for $k \geqq 2$ the theorem holds already for $k+r \geqq n$, and that this condition is sharp; a counter-example is given for $k+r=n-1$. Some even stronger continuation properties which are the analogue (for analytic sets) to the "Kontinuitätssatz" are also given.

{The reviewer conjectures: If G is r-convex, then G is the union of $n-r$ pseudo-convex domains, and the cohomology groups $H^q(G, \Omega)$ with coefficients in the sheaf of germs of holomorphic functions Ω vanish for $q \geqq n-r$.}

H. J. Bremermann (Seattle, Wash.).

★**Spencer, D. C. Potential theory and almost-complex manifolds.** Lectures on functions of a complex variable, pp. 15–43. The University of Michigan Press, Ann Arbor, 1955. $10.00.

A series of questions in the theory of external differential forms on complex-analytic manifolds is extended to almost-complex manifolds. The classification of forms according to type is set up. Operators ∂ and $\bar{\partial}$ are defined, analogous to $\sum dz_i\partial/\partial z_i$ and $\sum d\bar{z}_i\partial/\partial\bar{z}_i$ for the complex-analytic case, but, in distinction to that case, they are not in general differentials, since ∂^2 and $\bar{\partial}^2$ may be different from zero. The concept of an "almost Hermitian" metric on a manifold is introduced, and, by its means, a scalar product between forms is defined. The operators δ and $\bar{\delta}$ are considered, adjoint respectively to ∂ and $\bar{\partial}$ (with respect to the scalar product introduced), and also the Laplacian $\Delta=2(\delta\bar{\partial}+\bar{\partial}\delta)$. The author also introduces Green and Neumann operators and proves the solvability of the first boundary problem of potential theory for regions with compact closure. The Cauchy formula is noted for almost-analytic functions f (that is, such that $\bar{\partial}f=0$).

In conclusion the author touches upon manifolds admitting a complex-analytic structure (in that case $\bar{\partial}^2=\delta^2=0$). He also introduces the construction (Grothendieck) of the homotopy operator K, by means of which the form ϕ is locally represented as $\phi=(\bar{\partial}K+K\bar{\partial})\phi$. Also noted are the result of Dolbeault [C. R. Acad. Sci. Paris **236** (1953), 175–177, 2203–2205; MR **14**, 673, 1006] about $\bar{\partial}$-cohomologies of complex-analytic varieties and the result of Hirzebruch concerning the equality of the Todd genus and the arithmetic genus of an algebraic manifold. The question of generalizing these results to the almost-complex case is raised.

I. Z. Rozenknop (RŽMat **1957**, no. 7018).

Legrand, Gilles. Sur les variétés à structure de presque-produit complexe. C.R. Acad. Sci. Paris **242** (1956), 335–337.

The author gives a definition of almost-complex structure and exhibits several formulas concerning the torsion of the almost-complex structure. The definition given in this note is equivalent to the notion of quasi-complex manifold as found in Steenrod's "The topology of fibre bundles" [Princeton, 1951; MR **12**, 522]. The form of the definition in this note is suggested in D. C. Spencer's "Differentiable manifolds" [mimeographed notes, Princeton Univ.]. Further discussion of almost-complex manifolds may be found in D. C. Spencer's paper reviewed above.

In this note a torsion tensor is given for the almost-complex structure, the vanishing of which, at least if V^n is real analytic, is necessary and sufficient to insure that the almost-complex structure actually be complex analytic. Also obtained is an extension of the formula of Eckmann and Frölicher [Frölicher, Math. Ann. **129** (1955), 50–95, p. 60; MR **16**, 857]. *P. E. Conner* (Princeton, N.J.).

Algebraic Geometry

Bonera, Piero. Sopra alcune generalizzazioni della superficie desmica. Ist. Lombardo Sci. Lett. Rend. Cl. Sci. Mat. Nat. **91** (1957), 403–412.

La surface algébrique F ici considérée est le lieu d'un point de l'espace S_m à $m \geqq 3$ dimensions dont les coordonnées homogènes $x_1, x_2, \cdots, x_{m+1}$ sont telles que leurs puissances n-ièmes forment un groupe de nombres projectif à un groupe de nombres donnés $\lambda_1, \lambda_2, \cdots, \lambda_{m+1}$ tous différents entre eux. Il s'ensuit $x_i{}^n=(u_1-\lambda_i u_2)/(v_1-\lambda_i v_2)$, ou encore $x_i{}^n=(u-\lambda_i)/(v-\lambda_i)$ $(u=u_1/u_2,\ v=v_1/v_2)$. Si $n=1$ on obtient un cône rationnel $y_i=(u-\lambda_i)/(v-\lambda_i)$, d'ordre $m-1$, qui est représenté sur le plan par un système linéaire de courbes d'ordre $m+1$ ayant un point de base m-ple et $m+2$ points de base simples dont $m+1$ alignés. Si $n>1$ on a une surface unirationnelle qui s'obtient de la précédente à l'aide de la transformation bien connue $y_i=x_i{}^n$. La surface F contient $n^m(m+1)$ droites joignant les points fondamentaux des coordonnées avec les n^m points tels que $x_1{}^n=x_2{}^n=\cdots=x_{m+1}{}^n$. La surface F admet un groupe fini de collineations qui est mériédriquement isomorphe à celui des homographies du groupe de nombres donné $\lambda_1, \lambda_2, \cdots, \lambda_{m+1}$. *E. G. Togliatti* (Genova).

Sokolow, N. P. Affin-projektive Klassifikation der reellen ebenen Kurven dritter Ordnung. Ukrain. Mat. Ž. 9 (1957), 176–194. (Russian. German summary)

A classification of the cubics in a real affine plane based upon a complete system of 22 invariants. There are 66 irreducible and 44 reducible types of such cubics. Canonical forms of equations are given for all types.

F. J. Terpstra (Pretoria).

Nagata, Masayoshi. A Jacobian criterion of simple points. Illinois J. Math. 1 (1957), 427–432.

Zariski's generalized Jacobian criterion for simple points on an algebraic variety is here extended to the case of algebroid varieties, a previous extension by Samuel [Proc. Nat. Acad. Sci. U.S.A. 41 (1955), 647–650; MR 17, 300] being somewhat incomplete. The present proof is also valid in the algebraic case and in the case of rings of convergent power series over a field with an ordinary valuation. *M. Rosenlicht* (Evanston, Ill.).

Kodaira, K. Characteristic linear systems of complete continuous systems. Amer. J. Math. 78 (1956), 716–744.

In this paper the author establishes the following main theorem. Let C be a non-singular prime divisor belonging to a complete continuous system $\mathscr{M}$ of effective divisors on a non-singular (complex) projective algebraic variety V. If C satisfies the condition $H^1(V, \Omega(C))=0$, where $\Omega(C)$ denotes the sheaf of germs of meromorphic functions on V which are multiples of $-C$, then C corresponds to a simple point of the canonical parameter variety M of $\mathscr{M}$ and the characteristic linear system of $\mathscr{M}$ on C is complete. Moreover, the complete continuous system containing C is uniquely determined by C.

The proof proceeds on the following lines. The author shows first that if E is a sufficiently ample divisor on V the set $\mathscr{C}$ of all effective divisors homologous to E is parametrized by a fibre bundle Λ over the Picard variety $\mathfrak{P}$ of V. The fibre of Λ over a point P of $\mathfrak{P}$ is the projective space parametrizing the complete linear system $|E+P|$. The fact that all these projective spaces have the same dimension is needed as an essential step in the construction of Λ. An important auxiliary tool in this construction is a line bundle Ξ on $V\times\mathfrak{P}$ corresponding to the so-called Poincaré divisor. Ξ has the property that its restriction to any $V\times P$ (which we identify with V) is precisely the line bundle corresponding to P.

Now, given any effective divisor C, we take $E=C+L$, where L is a suitably ample hyperplane section of V. The set $\mathscr{N}$ of all effective divisors homologous to C is then parametrized by the subvariety N of Λ representing divisors of the form $C'+L$ with C' effective. The author shows that the analytic structure on N obtained in this way is unique and does not depend on L. This justifies the adjective "canonical" in the main theorem.

Suppose now that C satisfies the condition $H^1(V, \Omega(C))=0$. An analysis of the dimension of the system $|C+P|$ in the neighbourhood of $P=0$ shows that C is represented by a simple point $\mu(C)$ of N. Hence, C belongs to a unique complete continuous system $\mathscr{M}$ represented by the irreducible component M of N containing $\mu(C)$. If in addition C is a non-singular prime divisor, one can define the characteristic linear system of $\mathscr{M}$ on C using the tangent space to M at $\mu(C)$. The main theorem now follows easily from the precise information already obtained about the position of M as a subvariety of Λ.

M. F. Atiyah (Cambridge, England).

Severi, Francesco. Réflexions et problèmes sur la topologie des variétés algébriques. Schr. Forschungsinst. Math. 1 (1957), 235–244.

★ **Severi, Francesco. On the symbol of virtual intersection of algebraic varieties. Algebraic geometry and topology.** A symposium in honor of S. Lefschetz, pp. 157–166. Princeton University Press, Princeton, N. J., 1957. $7.50.

Summarizes the author's previous attempts to define rational equivalence, urging that these definitions are in fact precise. *M. Rosenlicht* (Evanston, Ill.).

★ **Zariski, Oscar. The connectedness theorem for birational transformations. Algebraic geometry and topology.** A symposium in honor of S. Lefschetz, pp. 182–188. Princeton University Press, Princeton, N. J., 1957. $7.50.

The result in point states that if V, V' are birationally equivalent (projective) varieties and $P\in V$ is a point at which V is analytically irreducible, then the set of points of V' that correspond to P is connected; this is a special case of a more general result which the author proved using his theory of holomorphic functions [Mem. Amer. Math. Soc. no. 5 (1951); MR 12, 853]. The present paper gives a more elementary proof under the assumption that P is simple on V. The result is trivial unless P is a fundamental point, in which case, if isolated, it corresponds to a locus of codimension 1 on V'. If one uses induction on dim V, the case in which P is not an isolated fundamental point descends easily by taking hypersurface sections through P, while the contrary case is reduced to this by the purely valuation-theoretic proof of the nonexistence of an infinite sequence $\{(V_i, P_i)\}$, $i>0$, $(V_1=V, P_1=P)$ such that each P_i is the center on V_i of a fixed divisorial valuation v of the function field of V and V_{i+1} is the quadratic transform of V_i out of P_i. *M. Rosenlicht.*

Chow, Wei-Liang; and Lang, Serge. On the birational equivalence of curves under specialization. Amer. J. Math. 79 (1957), 649–652.

Let Z be a cycle in a projective space S^n, rational over a field k with a discrete valuation; let $\mathfrak{p}$ be the maximal prime ideal in the valuation ring, and $\bar{Z}$ be the cycle obtained from Z by reduction modulo $\mathfrak{p}$ in the sense of Shimura. $\bar{Z}$ is called the specialization of Z (with respect to the given discrete valuation). Let C_1 and C_2 be two non-singular curves in S^n, F be a birational correspondence between C_1 and C_2 and assume that C_1, C_2, F are defined over k. The authors prove that if the specializations $\bar{C}_1$, $\bar{C}_2$ of C_1, C_2 are irreducible and non-singular, the specialization $\bar{F}$ of F is a birational correspondence between $\bar{C}_1$, $\bar{C}_2$, provided the genus of C_1, $C_2>0$. This generalizes the result of Deuring [Nachr. Akad. Wiss. Göttingen. Math.-Phys. Kl. IIa. 1955, 13–42; MR 17, 17] in the case of elliptic curves. First, the corresponding theorem is proved for Abelian varieties, which is done simply. Then the authors reduce the problem to the case of Jacobian varieties, which are constructed by the method of Chow. The theorem also generalizes the result of van der Waerden in the case of geometric specialization [Math. Ann. 114 (1937), 683–699], which is not mentioned in the paper.

T. Matsusaka (Evanston, Ill.).

Rosenlicht, Maxwell. Some rationality questions on algebraic groups. Ann. Mat. Pura Appl. (4) 43 (1957), 25–50.

The main result contained in this paper is the following.

Let G be a connected linear algebraic group defined over a perfect field k. Then the field $k(G)$ of rational functions on G defined over k is isomorphic to a subfield of a purely transcendental extension of k; this implies that, if k is infinite, then the set of points of G which are rational over k is dense in G. It is shown by examples that none of the assumptions in the statement of the theorem (connectedness of G, linearity of G, perfect character of k) may be dropped without the conclusion becoming false. It is also proved that, if G is a connected nilpotent linear algebraic group defined over an arbitrary field k, then the maximal torus of G is defined over k; however, an example shows that the group G_u of unipotent elements of G need not be defined over k, even if G is commutative. Finally, it is shown that, if G is a connected algebraic group defined over a field k, then the maximal connected linear subgroup of G may not be defined over k.

C. Chevalley (Paris).

Roth, Leonard. Further properties of pseudo-Abelian varieties. Rend. Sem. Mat. Univ. Padova **27** (1957), 1–15.

The following theorems are established. (1) Any algebraic variety which admits a finite continuous group of automorphisms and which is neither Picardian or pseudo-Abelian, is birationally equivalent to a ruled variety (i.e., one containing a congruence of linear spaces). (2) If an algebraic variety of this kind is such that the group leaves transitively invariant a linear system of positive dimension of hypersurfaces, which is not compounded of a congruence of subvarieties of positive dimension, then it is birationally equivalent to a ruled variety. (3) On any pseudo-Abelian variety W_p of type q $(1\leq q\leq p-1)$, the involution cut out by the congruence $\{V_q\}$ of trajectories on a member of the complementary congruence $\{V_{p-q}\}$ can be generated by a finite group of automorphisms, which is either cyclic or the product of k cyclic groups, $2\leq k\leq 2q$. *J. A. Todd.*

Kirby, David. Intorno alla classificazione dei rami e dei loro centri associati di proiezione. II. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) **20** (1956), 325–328.

This Note II is a continuation of Note I [same Rend. (8) **20** (1956), 179–184; MR **19**, 459] of the author concerning the study of the ring Γ of analytic elements passing through a given point O of projective space of a finite number of dimensions. Various algebraic properties of this ring Γ and its subrings are obtained. In particular, canonical rings and their associated canonical subgroups are studied. For example, the intersection of a collection of canonical rings is a canonical ring. Finally, basis representations of such a canonical ring are developed.

J. De Cicco (Chicago, Ill.).

Kirby, David. Intorno alla classificazione dei rami e dei loro centri associati di proiezione. III. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) **20** (1956), 446–451.

The author studies a certain type of projection T with a given center of projection P of an affine space A_n upon a proper hyperplane L. Such a projection T is independent of the chosen hyperplane L. By T, an analytic element Γ of A_n is carried into an analytic element Γ' situated in L. Such a Γ' is called a projection of Γ. The ring $H(t)$ generated by the analytic elements Γ is projected by such a T into a ring $H'(t)$ generated by Γ'. The conditions are obtained that an analytic element Γ' be obtained by such a projection T from an analytic element Γ. Applications are made to the set of analytic transformations T which leave the origin O invariant, by studying the affine part $L(T)$ of any such T. *J. De Cicco* (Chicago, Ill.).

See also: Statistics: Farchi.

NUMERICAL ANALYSIS

Numerical Methods

Moriguti, Sigeiti. On the rounding of Lagrangian interpolation coefficients. Rep. Statist. Appl. Res. Un. Jap. Sci. Engrs. **4** (1957), 147–152.

Rehwald, Walther; und Zinke, Otto. Anwendung einer Annäherung des Logarithmus durch ein Potenzgesetz auf elektrodynamische Probleme. Arch. Elek. Übertr. **11** (1957), 397–402.

Ward, L. E., Jr. On the optimal allocation of limited resources. Operations Res. **5** (1957), 815–819.

A computational algorithm (and numerical example) is given for the solution of the following concave programming problem, by reduction to a sequence of applications of Dantzig's simplex algorithm for linear programming problems. If $\chi_1, \cdots, \chi_n$ are non-negative integral variables; if $P_1, \cdots, P_n$ are disjoint subsets of the integers $1, \cdots, N$; and if σ_i denotes the sum of all χ_k such that k is a member of P_i; then the problem is to maximize $Z=\sum_{i=1}^{n}(g_i\sigma_i)$ subject to the restrictions: (a) $0\leq\chi_k\leq M_k$ for $k=1, \cdots, N$; (b) $\sum_{k=1}^{n} a_{jk}\chi_k\leq A_j$ for $j=1, \cdots, m$; and (c) $\sum_{k=1}^{n} b_{jk}\chi_k$ is an integral multiple of α, each $b_{jk}=0$ or 1, for $j=1, \cdots, m$. *M. M. Flood.*

Bellman, Richard. On the computational solution of linear programming problems involving almost-block-diagonal matrices. Management Sci. **3** (1957), 403–406.

Sabroff, Richard R.; and Higgins, T. J. A critical study of Kron's method of "tearing". II. Matrix Tensor Quart. **8** (1957), 5–12.

See the review of part I in MR **19**, 64.

Panc, Vladimír. Die verbesserte Relaxationsmethode. Apl. Mat. **2** (1957), 184–201. (Czech. Russian and German summaries)

Es handelt sich um eine Verbindung der zuerst von V. Dašek und R. V. Southwell entwickelten Relaxationsmethoden [cf., e.g., R. V. Southwell, Relaxation methods in engineering science, Oxford, 1940; MR **3**, 152] die sich eine Beschleunigung der auftretenden Konvergenzprozesse zum Ziel setzt. Dafür gewinnt Verfasser zwei Methoden, deren Gruppenoperationen sich von der Southwellschen Methode unterscheiden und unter Verwendung von Dreiecksmatrizen zu einer Relaxationselimination führen, welche das bekannte Gaußsche Verfahren praktisch verbessert. Nach einer theoretischen Behandlung des Systems

$$x_1+\sum_{k=2}^{n} a_{1k}x_k=a_{10},\quad x_2+\sum_{2\neq k=1}^{n} a_{2k}x_k=a_{20},\ \cdots,$$
$$x_n+\sum_{k=1}^{n-1} a_{nk}x_k=a_{n0}$$

werden zwei numerische Beispiele durchgerechnet. Dabei wird noch insbesondere auf die Umstände hingewiesen, welche die Anwendung der entwickelten Methoden konvergenzbeschleunigend beeinflussen können. *M. Pinl.*

Miroux, Jean. Une nouvelle machine analogique à itération matricielle donnant les racines des équations algébriques. L'itération des matrices ayant des valeurs propres de modules voisins. Ann. Télécommun. **11** (1956), 226–232.

This paper describes how to adapt the usual power method for finding the latent roots of matrices when there is a cluster of dominant roots with close absolute values. It presents a general method for deriving from successive iterates the algebraic equation satisfied by these roots, and applies the method to a number of special cases, real double roots, conjugate pure imaginary roots, pairs of such roots, and complex roots with small imaginary parts. The application of the method for calculating the roots of an algebraic equation by considering the corresponding matrix is elaborated and an analogue machine based on this method is described. It can be used to find the roots, real and complex, of an algebraic equation with real coefficients. The number of elements needed in the machine is proportional to the degree of the equation, and it is claimed that accuracy is not affected by the order. *J. H. Wilkinson.*

Eisemann, Kurt. Removal of ill-conditioning for matrices. Quart. Appl. Math. **15** (1957), 225–230.

L'auteur propose une méthode pour traiter des systèmes d'équations linéaires instables ("ill-conditioned systems"), dans le seul cas où quelques-unes des équations sont presque linéairement dépendantes; la méthode ne s'applique pas aux systèmes, dans lesquels des équations en nombre quelconque sont presque linéairement dépendantes.

Dans ce but, l'auteur se sert d'abord de la méthode d'élimination qu'il arrête lorsque l'une des équations obtenues, soit E' (issue d'une équation E du système initial) présente des coefficients qui sont tous en valeur absolue inférieurs à une limite fixée (0.20 dans l'exemple numérique traité). Un système triangulaire formé à partir de coefficients déjà calculés permet d'obtenir des valeurs approchées des coefficients qui donnent directement E' à partir de E et des équations initiales qui ont fourni les équations-pivot; cette combinaison linéaire, qui donnerait en principe E', est reprise par un calcul en double précision qui conduit à une équation E^*, dont les coefficients d'abord petits sont multipliés par une puissance de 10 convenable. A l'équation E^* ainsi modifiée on applique la méthode d'élimination en utilisant les équations-pivot déjà obtenues, ce qui fournit finalement, à la place de E', une équation dont tous les coefficients sont supérieurs en valeur absolue à la limite fixée, et qui remplace E' dans la suite des calculs.

Après la justification théorique, l'auteur donne un exemple numérique de 5 équations à 5 inconnues et compare ses résultats à ceux qu'aurait donnés une élimination poursuivie sans précautions; le gain de précision ainsi obtenu est considérable. *G. Brillouet* (Nantes).

Kosko, Eryk. Matrix inversion by partitioning. Aero. Quart. **8** (1957), 157–184.

A systematic discussion, with practical procedures and applications. A square matrix partitioned into a number of sub-matrices is called a super-matrix, and the discussion in based on this concept. Sections deal with partitioning into 2×2 sub-matrices, inverting an augmented matrix, 3×3 partitioning, and more complicated inversion problems. *W. F. Freiberger* (Providence, R.I.).

Sobol', I. M. On an iterative method for calculation of eigenvalues. Uspehi Mat. Nauk (N.S.) **12** (1957), no. 3(75), 377–380. (Russian)

Let $p(x)$, $p'(x)$, $q(x)$, $w(x)$ be continuous for $0\leq x\leq 1$, and let $p(x)>0$, $w(x)>0$. Denote by $u(x, \lambda)$ the solution of

$$(pu')'-qu+\lambda wu=0$$

satisfying $u(0, \lambda)=0$, $\int_0^1 w(x)u^2(x, \lambda)dx=1$. Define

$$D(\lambda)=\int_0^1 [p(x)u'^2(x, \lambda)+q(x)u^2(x, \lambda)]dx.$$

Using the fact that $u'(1, \lambda)=0$ implies $\lambda=D(\lambda)$, V. J. Berry and C. R. de Prima [J. Appl. Phys. **23** (1952), 195–198; MR **13**, 782] proposed to calculate the eigenvalues μ_n corresponding to the boundary conditions $u(0)=u'(1)=0$ by solving the equation $\lambda=D(\lambda)$ by iteration, and stated that the procedure would always converge. The argument was based on the monotonicity of the function $D(\lambda)$, which, however, was not fully proved. It is now shown in the paper under review that $D(\lambda)$ fails to be monotonic even for $y''+\lambda y=0$. The author then tries to reestablish the Berry-de Prima procedure by imposing the additional condition that $\lambda_{n-1}<\lambda<\lambda_n$ implies $\lambda_{n-1}<D(\lambda)<\lambda_n$ (the λ_n are the eigenvalues corresponding to $u(0)=u(1)=0$). In the opinion of the reviewer the author's argument is fallacious, since a limit point of the sequence $\{\lambda^{(\nu)}\}$ generated by $\lambda^{(\nu+1)}=D(\lambda^{(\nu)})$ is not necessarily a solution of $\lambda=D(\lambda)$, as asserted. *P. Henrici.*

Bodewig, E. Zum Matrizenkalkül. V. Nederl. Akad. Wetensch. Proc. Ser. A. **60**=Indag. Math. **19** (1957), 242–247.

Let $f(z)=\sum_{k=0}^{\infty} s_k z^{-k}$ be a Laurent expansion of a rational function with poles $\lambda_1, \cdots, \lambda_n$. We assume that $|\lambda_1|>|\lambda_2|>\cdots>|\lambda_n|$. The Quotient-Difference (Q-D) algorithm of H. Rutishauser [Z. Angew. Math. Phys. **5** (1954), 233–251, 496–508; **6** (1955), 387–401; MR **16**, 176, 863; **17**, 789] in its simplest form produces numbers $q_m^{(k)}$ $(m=1, 2, \cdots, n;\ k=0, 1, 2, \cdots)$ which depend in a non-linear manner on $s_k, \cdots, s_{k+2m-1}$ and satisfy $q_m^{(k)}\to\lambda_m$ $(k\to\infty)$. This amounts to the simultaneous determination of all poles of $f(z)$. The author of the article under review takes issue with this algorithm. He proposes to solve the same problem in what he calls a simpler and speedier manner as follows: (1) Determine $\lambda_1=\lim s_{k+1}/s_k$. (2) Form the coefficients $t_k=s_{k+1}-\lambda_1 s_k$. The function $g(z)=\sum t_k z^{-k}$ has λ_2 as its largest pole. Application of (1) to $g(z)$ will yield λ_2, by (2) λ_2 is eliminated, and so on. The problems of speeding up convergence in the case of multiple poles and of treating poles with equal moduli are also discussed. {Reviewer's remarks: Both the Q-D algorithm (as described above) and the author's algorithm are numerically unstable and will not produce good values for λ_s unless s is very small. For the Q-D algorithm this was emphasized by H. Rutishauser (loc. cit.); for the author's algorithm, instability was verified in unpublished numerical experiments performed by the reviewer in 1955. The

author seems to overlook the fact that the Q-D algorithm can be rendered stable if performed in the so-called "progressive" form, to which most of Rutishauser's work has been devoted. There is no analog to the progressive form for the author's algorithm, because successive columns cannot be generated unless all previous λ_i are known.} *P. Henrici* (Los Angeles, Calif.).

Troitskaya, E. A. Application of the general theory of approximation methods to the problem of determination of characteristic numbers and characteristic vectors. Dokl. Akad. Nauk SSSR (N.S.) **113** (1957), 998–1001. (Russian)

Let X and $\bar{X}$ be two Banach spaces. It is assumed that a subspace $\tilde{X}$ of X is isomorphic to $\bar{X}$. The isomorphism is realized by a linear operator φ_0 with domain $\tilde{X}$ and range $\bar{X}$ and possessing an extension φ with domain X. Consider two bounded linear operators A and $\bar{A}$, with domains X and $\bar{X}$, satisfying these conditions: (1) $\|\varphi A\tilde{x}-\bar{A}\varphi\tilde{x}\|\leq\epsilon\|\tilde{x}\|$, $\tilde{x}\in\tilde{X}$; (2) given $x\in X$, there exists $\tilde{x}\in\tilde{X}$ such that $\|Ax-\tilde{x}\|\leq\epsilon_1\|x\|$. Let λ_0 be a simple characteristic value of A, let x_0 be the corresponding characteristic element, and let f_0 be the corresponding characteristic element of A^*, the adjoint of A, normalized by $f_0(x_0)=1$. It is observed that the pair $\begin{pmatrix}x_0\\ \lambda_0\end{pmatrix}$ is a solution of the nonlinear functional equation

$$P\begin{pmatrix}x\\ \lambda\end{pmatrix}=\begin{pmatrix}Ax-\lambda x\\ f_0(x)-1\end{pmatrix}=0.$$

It is proposed to solve the "neighboring" characteristic value problem in the space $\bar{X}$ by solving the corresponding equation

$$(*)\qquad \bar{P}\begin{pmatrix}\tilde{x}\\ \lambda\end{pmatrix}=\begin{pmatrix}\bar{A}\tilde{x}-\lambda\tilde{x}\\ f_0(\varphi_0^{-1}\tilde{x})-1\end{pmatrix}=0$$

by a generalization of Newton's method due to L. V. Kantorovič [Uspehi Mat. Nauk (N.S.) **3** (1948), no. 6(28), 89–185; MR **10**, 380], using the element $\begin{pmatrix}\varphi x_0\\ \lambda_0\end{pmatrix}$ as a first approximation. Utilizing the general theory of approximate processes due to Kantorovič (loc. cit.), two theorems concerning the existence of a solution of (*) and its closeness to the initial approximation in terms of ϵ, ϵ_1, and the norms of A, $\bar{A}$, φ are given. *P. Henrici.*

Fettis, H. E. On the eigenvalues of Latzko's differential equation. Z. Angew. Math. Mech. **37** (1957), 398–399.

Consider the differential equation

$$(1)\qquad \frac{d}{dx}(1-x^7)\frac{dy}{dx}=-\omega x^7 y \qquad (0\leq x\leq 1)$$

with the boundary conditions $y(0)=0$ and $y(1)$ finite. H. Latzko [Z. Angew. Math. Mech. **1** (1921), 268–290] determined crude approximations for the first three eigenvalues of (1) using Legendre polynomials and, more recently, W. H. Durfee [J. Aero. Sci. **23** (1956), 188–189] has obtained more accurate values by direct series expansion of the solution of (1). Using Jacobi polynomials, the present author obtains good estimates for the first three eigenvalues and eigenfunctions with much fewer terms than are needed for Legendre polynomials.

J. K. Hale (St. Paul, Minn.).

Rutishauser, Heinz. Solution of eigenvalue problems with the *LR*-transformation. Nat. Bur. Standards Appl. Math. Ser. no. **49** (1958), 47–81.

This paper gives a theoretical and numerical treatment of the so-called *LR*-transformation as applied to the determination of latent roots and vectors of matrices. The triangular decomposition $A_1=L_1R_1$, where L_1 is lower diagonal with unit terms and R_1 is upper triangular, is applied to the given matrix $A=A_1$. Then A_2 is defined as R_1L_1 and A_2 is decomposed into L_2R_2. The process continues. The resulting matrices A_k are similar and hence have the same latent roots as A. The L matrices and the R matrices can be multiplied respectively to give $\Lambda_k=L_1L_2\cdots L_k$ and $P_k=R_kR_{k-1}\cdots R_1$. It is shown that $A^k=\Lambda_kP_k$ and that if the matrices Λ_k converge for $k\to\infty$, then $\lim_{k\to\infty}A_k$ exists and is an upper triangular matrix, A_∞, whose diagonal terms are the latent roots of A. Different conditions for the convergence of Λ_k are given. One of these is that A be Hermitian and positive definite. Numerical examples are introduced early and are used throughout the paper. The case of a real matrix with complex latent roots is featured.

It is admitted that the process is time consuming, but it is simple and allows easy checks. The process seems especially applicable to "striped" matrices, with $a_{ij}=0$ for $|i-j|>m$, since each A_k retains the striped property with the same m. For the general problem, a modification in the later steps of the procedure is introduced to improve the convergence and is applied to the case of real latent roots and to that of conjugate complex latent roots. A section is devoted to the determination of latent vectors, another to corrective measures against round off errors, another on the use of "deflation", and another to striped matrices. A final section features a continuous analogue to the *LR*-transformation. There is a bibliography of 18 references. *P. S. Dwyer* (Ann Arbor, Mich.).

Clendenin, W. W. Modified relaxation method for eigenvalue problems. J. Math. Phys. **36** (1957), 74–81.

A method for the solution of the one-dimensional Schroedinger equation

$$d^2\psi/dx^2+[\Lambda+U(x)]\psi=0,$$

where $U(x)$ is defined over the region $a\leq x\leq b$ and Λ is an eigenvalue, is obtained by minimising the Ritz integral with respect to the net points ψ_n as variational parameters. A quadratic interpolation formula for the ψ_n is used. Singularities in $U(x)$ of the form $A/(x-a)$, $B/(x-a)^2$ are treated by considering separately the intervals $a\leq x\leq a+h$, $a+h\leq x\leq a+2h$. The method gives identical formulae, except at the end-points, where it seems to be inferior, with those obtained by direct substitution of the difference formulae;

$$h^2d^2\psi/dx^2=\delta^2\psi_0-\tfrac{1}{12}\delta^4\psi_0+\tfrac{1}{90}\delta^6\psi_0-\cdots,$$

$$h^2d^2\psi/dx^2=\delta^2\psi_0-\tfrac{1}{12}\delta^4\psi_1+\tfrac{1}{12}\delta^5\psi_{1/2}-\tfrac{13}{180}\delta^6\psi_2+\cdots,$$

retaining terms only up to $\delta^4\psi_0$. Apparently the author recommends that in the solution of these equations by the relaxation methods, the fourth difference terms should be included in the relaxation patterns instead of being treated separately as suggested by Fox [Proc. Roy. Soc. London. Ser. A. **190** (1947), 31–59; MR **9**, 106].

D. C. Gilles (Glasgow).

Schlechtweg, H. Zur Abschätzung des Restgliedes der Mittelwertformeln zur genäherten Quadratur. Z. Angew. Math. Mech. **37** (1957), 353–361. (English, French and Russian summaries)

In this paper the author derives remainders for approximate quadrature formulas of the form

$$\int_0^1 f(x)dx \approx \sum_{v=1}^{n} A_v f(x_v),$$

where the remainders have the form

$$R_n = \sum_{v=1}^{n} A_v \int_0^1 dx f'(x)\bar{B}_1(x_v - x),$$

and $\bar{B}_1(x)$ is the first periodic Bernoulli polynomial, defined by $\bar{B}_1(x)=x-\frac{1}{2}$, $0 \leq x < 1$, and $\bar{B}_1(x+1)=\bar{B}_1(x)$.

These formulas differ from the usual ones by involving the first derivative only. Tables to implement the use of some of the formulas are given. *P. C. Hammer.*

Fettis, Henry E. Lommel-type integrals involving three Bessel functions. J. Math. Phys. **36** (1957), 88–95.

The author considers the problem of numerically evaluating integrals of the form

$$I(\alpha, \beta, \gamma, x) = \int_0^x t J_0(\alpha t) J_0(\beta t) J_0(\gamma t) dt.$$

These values are needed in certain problems involving Fourier-Bessel series of the form $\sum_{k=1}^{\infty} A_k J_0(\alpha_k x)$.

The problem is solved in two ways. First, after the integral has been reduced to a combination of two integrals involving only one Bessel function in each, these integrals are expanded in Neumann series and error bounds for truncation are given. This is reported to be useful for hand computation when α, β, γ are real. The other mode of approximations involves using the Poisson summation formula which is the trapezodial rule with a correction series. Again the error formula is given, and an example to show the effectiveness of this quadrature formula is calculated.

A table of values of $I(\lambda_i, \lambda_j, \lambda_k, 1)$ is given when $i, j, k = 1, 2, 3$ and $\lambda_1, \lambda_2, \lambda_3$, are the first three zeros of $J_0(x)$. One misprint occurs on page 90, formula (12), in which $G_2(\alpha, \beta, \gamma, x)$ should read $G_1(\alpha, \beta, \gamma, x)$.

P. C. Hammer (Madison, Wis.).

Peirce, William H. Numerical integration over the planar annulus. J. Soc. Indust. Appl. Math. **5** (1957), 66–73.

The classical Gauss quadrature formula for the integral

$$\int_a^b f(x)dx$$

is extended to the case of the double integral

$$I = \int f(x, y)dxdy = \int_R^1 \int_0^{2\pi} rF(r, \theta)d\theta dr,$$

where $0 \leq R < 1$ and $F(r, \theta) = f(r\cos\theta, r\sin\theta)$. This integral over the annulus $R < r < 1$ is approximated by the double sum

$$(*) \qquad \sum_{i,j} D_{ij} F(r_j, \theta_i).$$

The author considers the case in which the θ_i are equally spaced and seeks the best choice of r_j so that the error in the approximation (*) vanishes whenever $f(x, y)$ is a polynomial of degree at most k in x and y. When $k = 4m+3$ it is necessary and sufficient to choose

$$\theta_i = 2\pi i/(k+1) \quad i = 1(1)k+1$$

$$P_{m+1}(r_j^2) = 0$$

$$D_{ij} = C_j = \{4(m+1)P_{m+1}(r_j^2)\}^{-1} \int_{R^2}^{1} \frac{P_{m+1}(r^2)}{r^2 - r_j^2} d(r^2),$$

where $P_{m+1}(t)$ is the Legendre polynomial of degree $m+1$.

Two examples are given with $R=0$ and $m=1$ and $m=3$, which correspond to 16 and 64 interior points respectively. Nine functions are integrated over the unit circle, using both formulas, and percentage errors are given. *D. H. Lehmer* (Berkeley, Calif.).

Salzer, Herbert E. Equally-weighted quadrature formulas for inversion integrals. Math. Tables Aids Comput. **11** (1957), 197–200.

Using the same methods as in his previous paper on equally-weighted quadrature formulas for evaluating Laplace transforms [J. Math. Phys. **34** (1955), 54–63; MR **16**, 1055], the author develops similar formulas for evaluating inversion integrals of the form,

$$(1) \qquad \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \frac{e^p}{p} F(p)dp.$$

Quadrature formulas of the form,

$$(2) \qquad (1/n) \sum_{j=1}^{n} F(p_j),$$

are obtained for $n = 1(1)10$. For each n, a set of p_j is determined so that (2) gives the exact value of (1) whenever $F(p)$ is a polynomial in $z = 1/p$, of degree not greater than n. A necessary and sufficient condition for this property is easily obtained as

$$(3) \qquad \sum_{j=1}^{n} (1/p_j)^r = n/r! \quad (r = 1, \cdots, n).$$

Using (3), the coefficients a_k of the monic polynomial $\phi_n(z)$ having $z_j = 1/p_j$ as its zeros are calculated from the well-known recursion relations,

$$ka_k + a_{k-1}\sum_{j=1}^{n} z_j + a_{k-2}\sum_{j=1}^{n} z_j^2 + \cdots + a_1 \sum_{j=1}^{n} z_j^{k-1} + \sum_{j=1}^{n} z_j^k = 0, \quad k = 1(1)n.$$

The z_j are generally complex and are computed as the roots of $\phi_n(z) = 0$ on a digital computer. Both p_j and z_j are tabulated for $n = 1(1)10$. *E. K. Blum.*

Ceschino, Francis. L'intégration approchée des équations différentielles. C. R. Acad. Sci. Paris **243** (1956), 1478–1479.

A method for numerical integration of an ordinary first-order differential equation $y' = f(x, y)$ is given. The marching procedure is based on the Taylor expansion, and it starts from a formula which gives y_{i+1} of order p and then leads to improved values of y_{i+1} of order $p+1$, $p+2$, $\cdots$, $2p$ successively. For every step, it determines an optimal value of the interval $h_i = x_{i+1} - x_i$, which is valuable especially if the solution has large variations. The method is described as convenient for automatic calculations. *P.-O. Löwdin* (Uppsala).

Sauliev, V. K. On the numerical solution of a boundary problem for a system of ordinary differential equations. Dokl. Akad. Nauk SSSR (N.S.) **112** (1957), 1002–1004. (Russian)

Let $0 = r_0 < r_1 < \cdots < r_s = R$, and let A, B, C be three symmetric $n \times n$ matrices which are defined as functions of x in each of the intervals $I_k = (r_{k-1}, r_k)$ $(k = 1, \cdots, s)$ and satisfy the following conditions in each I_k: A is a constant positive definite diagonal matrix; B and C are twice continuously differentiable; C is positive definite uniformly in x. The author considers the following

characteristic value problem for a vector function $u=u(x)$ $(\prime=d/dx)$:

$$Au''+Bu+\lambda Cu=0,\ x\in I_k,\ k=1, 2, \cdots, s;$$
$$u(0)=u(R)=0;$$
$$u(r_k-0)=u(r_k+0)=u(r_k)$$
$$A(r_k+0)u'(r_k+0)-A(r_k-0)u'(r_k-0)$$
$$=r_k^{-1}(A(r_k+0)-A(r_k-0))u(r_k),\quad k=1, 2, \cdots, s-1.$$

He replaces the continuous problem in a standard manner by a finite difference problem with step h and asserts that for sufficiently small h the eigenvalues of smallest modulus λ and $\lambda^{(h)}$ of the two problems satisfy $|\lambda-\lambda^{(h)}|<ch^2$, where c is independent of h. For $n=1, s=1$, this reduces to a result of L. Collatz [Deutsche Math. 2 (1937), 189–215]. A proof, based on certain minimum properties of λ and $\lambda^{(h)}$, is outlined. *P. Henrici* (Los Angeles, Calif.).

Sabliet, S. Méthodes mathématiques utilisées pour résoudre numériquement les intégrales et les systèmes d'équations differentielles à l'aide des ordinateurs électroniques. Rev. Gén. Sci. Pures Appl. **64** (1957), 223–227.

Integration methods involving the use of the trapezoidal and Simpson's rule are presented for integrating $F'(x)$ on an electronic computer. There is an elementary discussion of the relative errors for these methods, but no mention of the effect of interval size. To solve the pair of equations

$$\frac{dg}{dt}=f(g, \Phi),\ \alpha\Phi+\beta\Phi\int_0^t\int_0^t \Phi g\,dt\,dt=\gamma,$$

for Φ and g, the second equation is differentiated twice, and an integration scheme using the trapezoidal rule is set up. By choosing g as the independent variable, and keeping Δg sufficiently small, it is stated that errors can be controlled. There are misprints on p. 224, l. 27, p. 225, l. 7, and p. 226, last line. *C. C. Gollieb* (Toronto, Ont.).

Farrington, C. C.; Gregory, R. T.; and Taub, A. H. On the numerical solution of Sturm-Liouville differential equations. Math. Tables Aids Comput. **11** (1957), 131–150.

The paper is concerned with the numerical solution of (1) $y''+[q(x)+\lambda r(x)]y=0$ with boundary conditions $y(a)=y(b)=0$ and normalization condition $\int_a^b ry^2dx=1$. Two methods are proposed, both based on the fact that (1) is the Euler condition for $\delta I(y)=0$, $I(y)=\int_a^b[(y')^2-qy^2]dx-\lambda\int_a^b ry^2dx$. Let $x_i=a+ih$, $h=(b-a)/(N+1)$, $y_i=y(x_i)$. In their first method, the authors approximate $I(y)$ by a quadratic form in y_i, using a quadrature formula $\int_a^b\Phi(x)dx\approx\sum_{-1}^{N+1}w_i\Phi(x_i)$ and a finite difference approximation $y'(x_i)\approx h^{-1}\sum_{-1}^{N+1}a_{ij}y_j$. The extremum condition then translates into an algebraic eigenvalue problem (2) $TY+\lambda^*RY=0$ with $T=(t_{jk})$ a symmetric matrix, R a diagonal matrix and Y a column vector. To determine the values of w_i, a_{ij} the authors require that T approximate the differential operator d^2/dx^2+q to a given order σ in the sense $\sum_{k=1}^N t_{jk}y_k=y''(x_j)+q(x_j)y_j+O(h^\sigma)$ $(j=1, 2, \cdots, N)$. General conditions for w_i, a_{ij} are derived and explicit solutions given for the cases $\sigma=1$ and $\sigma=2$. The second method is a variant of the Rayleigh-Ritz method, the novelty lying in the choice of the class of admissible functions. They are made up by appropriate interpolating polynomials defined separately in each interval $x_i\leq x\leq x_{i+1}$. The method leads to a problem of type (2) with symmetric matrices T and R. Explicit computational procedures are given for evaluating the elements of T and R. There is a report and discussion of numerical results for the simple example $y''+\lambda y=0$, $y(0)=y(1)=0$.

It is stated that both methods are readily adapted to more general boundary conditions of the form $\alpha_1y(a)+\alpha_2y'(a)=0$, $\beta_1y(b)+\beta_2y'(b)=0$. {It seems to the reviewer that $I(y)$ then should be supplemented by certain boundary expressions.} *Walter Gautschi.*

Greenspan, Donald. On the numerical evaluation of the Stokes' stream function. Math. Tables Aids Comput. **11** (1957), 150–160.

The author considers the solution by finite difference methods of boundary value problems associated with the equation $u_{xx}-(1/y)u_y+u_{yy}=0$. In deriving the difference equation $L(u)=\sum\alpha_iu_i$, the sum being taken over a suitable set of points of the net, the author chooses the coefficients α_i so that $L(u)$ will vanish to as high a power as possible in the mesh size h. (Actually, some additional conditions should be imposed on the α_i to eliminate the case where they all vanish and where, as a result, $L(u)$ vanishes also.) In working with the Taylor series expansion for $L(u)$, relations between the partial derivatives of u are obtained not only from the differential equation itself, but also from certain additional equations obtained by differentiating the differential equation. Moreover, the terms in the Taylor series of $L(u)$ are required to vanish only to an appropriate power of h. The method is used to derive two frequently used difference equation representations for the differential equation.

The methods of Gerschgorin [Z. Angew. Math. Mech. **10** (1930), 373–382] and Collatz [ibid. **13** (1933), 56–57] are applied to the particular differential equation to obtain bounds on the difference between the solution of the difference equation and the solution of the differential equation. These bounds involve the maximum of the absolute values of the partial derivatives of u of third, fourth, and fifth order. *D. M. Young.*

Varga, Richard S. A comparison of the successive overrelaxation method and semi-iterative methods using Chebyshev polynomials. J. Soc. Indust. Appl. Math. **5** (1957), 39–46.

If the equation $u_{i+1}=Mu_i+c$ $(i=0, 1, 2, \cdots)$, where M is a specific matrix, defines an iterative method for solving a linear system $Au=k$, then the sequence of vectors $t_n=\sum_{j=0}^n\nu_j(n)u_j$ $(n=0, 1, 2, \cdots)$, where the constants $\nu_j(n)$ are real numbers, defines a "semi-iterative method" with respect to the method defined by M. Assuming A is symmetric and positive definite and satisfies Property (A) in the sense of Young [Trans. Amer. Math. Soc. **76** (1954), 92–111; MR **15**, 562], the author shows that the successive overrelaxation method converges at least as fast as any semi-iterative method associated with the Jacobi method, the Gauss-Seidel method, or the successive overrelaxation method itself. In the latter case, the result is proved under the assumption that all eigenvalues of the matrix corresponding to the successive overrelaxation method lie in the circle $|\lambda|\leq\rho<1$. For the Jacobi method a proof is given which is different from that due to Young [Proc. Symposia Appl. Math., vol. VI, McGraw-Hill, New York, 1956, pp. 285–298; MR **18**, 417]. The author's proof generalizes to include the Gauss-Seidel method. The fact that the successive overrelaxation method, in which only the latest

iterate is needed at any stage, compares favorably with semi-iterative methods is fortunate from the standpoint of high-speed computing machine calculation since for the latter methods the simultaneous storage of several iterates is required.

Correction: The constant vector in the right member of (2) need not be k as defined in (1). *D. M. Young, Jr.*

Eichelbrenner, Ernest A. Conditions aux limites discontinues dans le cas d'écoulements en fluide visqueux. C. R. Acad. Sci. Paris **244** (1957), 2476–2479.

This paper describes a finite difference method for solving the initial-boundary value problem for the nonlinear parabolic equation $uu_x=u_{yy}$. The new unknown $v=1-u$ is introduced: $v_x-vv_x=v_{yy}$. The first term on the left is replaced by a forward difference, the second term — somewhat peculiarly — by a centered difference, and the term on the right is centered. The calculation proceeds stepwise in x; the new value of v is determined by an iteration; it seems to the reviewer that it can be determined explicitly. The author states that the difference equations are stable for any ratio of $\Delta x/\Delta y^2$; the reviewer found that the Courant-Friedrichs-Lewy condition is still necessary.

Some numerical examples are presented and used in boundary layer theory. *P. D. Lax.*

Aufschläger, Rudolf. Konvergenzuntersuchungen zur Massauschen Gitterkonstruktion bei Anfangswertproblemen partieller Differentialgleichungen. Bayer. Akad. Wiss. Math.-Nat. Kl. S.-B. **1956**, 87–112 (1957).

This paper describes an application of Massau's method of employing a characteristic lattice for the stepwise construction of solutions to initial value problems for nonlinear hyperbolic systems of three equations for three unknowns (such as the equation of compressible anisentropic flow). The lattice used is the one formed by the outside characteristics. The needed values along the middle characteristic are supplied by interpolation. This makes the truncation error of second order, which cannot be improved by trying to locate the lattice more accurately, as the author does by an iteration scheme.

There are convergence proofs, but no numerical examples. The exposition and notation are models of clarity. *P. D. Lax* (New York, N.Y.).

Skvorcov, V. S. Application of the method of nets to the solution of systems of partial differential equations. Dokl. Akad. Nauk SSSR (N.S.) **112** (1957), 20–23. (Russian)

Partial difference equations are considered which arise from partial differential equations when derivatives are replaced by differences. Following the method of Duffin [Duke Math. J. **20** (1953), 233–251; MR **16**, 1119], the fundamental matrix for the boundary value problem of the first kind is constructed using multiple finite Fourier transforms. Theorems are given evaluating the asymptotic behaviour of the fundamental matrix and the deviation of the fundamental matrix of the difference equation from the fundamental matrix of the corresponding partial differential equation. *C. Saltzer* (Cleveland, Ohio).

Filippov, A. F. On the application of the method of finite differences to the solution of the problem of Tricomi. Izv. Akad. Nauk SSSR. Ser. Mat. **21** (1957), 73–88. (Russian)

A finite difference formulation of the Tricomi problem is considered. The uniqueness and the existence of the solution of the finite difference system of equations are established, and it is shown that the solution can be approximated by the Gauss-Seidel iterative process. Theorems concerning the convergence of the solution of the difference equation to the solution of the Tricomi problem as the mesh size approaches zero are given which are based on the Taylor expansion. *C. Saltzer.*

Evans, G. W.; Brousseau, R.; and Keirstead, R. Stability considerations for various difference equations derived for the linear heat conduction equation. J. Math. Phys. **34** (1956), 267–285.

In the numerical solution of partial differential equations using finite-difference methods, a distinction is made between "convergence" and "stability" for problems of stopping-ahead type [see O'Brien, Hyman, Kaplan, same J. **29** (1951), 223–251; MR **12**, 751]. However, convergence and stability often go hand-in-hand. Stability is concerned with the behavior of errors entering the numerical solution of the difference equation; for practical computations, an analytic criterion for stability was developed by von Neumann and amplified by the reviewer. It is possible to manufacture examples, however, in which error behavior (i.e., stability) is anomalous and the criterion, while correct, does not give enough information. The authors give an interesting example of this sort, based on the heat equation (*) $u_t=u_{xx}$. They also discuss various iterative schemes for solving the usual implicit difference approximant to (*); these schemes amount to point relaxation; the authors suggest their use in higher dimensions. Unfortunately, there are several grave errors in the paper; see the following review.

M. A. Hyman (Ossining, N.Y.).

Evans, G. W.; Brousseau, R.; and Keirstead, R. Errata for "Stability considerations for various difference equations derived for the linear heat conduction equation". J. Math. Phys. **36** (1957), 294–295.

Here, the authors of the preceding paper have corrected various errors, in particular some typographical errors, a statement of convergence for a difference equation, and convergence conditions for a certain geometric series. Ignoring residual typographical errors, the reviewer notes three other lapses. On pp. 271–273, the example fails because the difference solution does not, as stated, converge to the differential solution. Also equation (53), p. 282, is certainly unstable for $\theta=0$, $r\equiv\Delta t/\Delta x^2>1$ and for other θ, r combinations. In the footnote to p. 271, the last word should be "stability" rather than "convergence". *M. A. Hyman* (Ossining, N.Y.).

Conte, S. D.; and Royster, W. C. A study of finite difference approximations to a fourth order parabolic differential equation. Ballistic Research Laboratories, Aberdeen Proving Ground, Md., Rep. no. **959** (1955), 22 pp.

The authors consider numerical solution by difference techniques of $u_{tt}+u_{xxxx}=0$. After some discussion of stability and convergence, they propose the implicit scheme

$$(*)\qquad \Delta_t{}^2u(x,t)+\tfrac{1}{4}[\Delta_x{}^4u(x,t-\Delta t)+2\Delta_x{}^4u(x,t)+\Delta_x{}^4u(x,t+\Delta t)]=0;$$

here $\Delta_t{}^2u(x,t)$, $\Delta_x{}^4u(x,t+\Delta t)$ are divided central differences in t and x, about the respective points (x,t) and $(x,t+\Delta t)$. The authors solve (*) using an LU-factorization

of the 5-diagonal matrices occurring; they comment on the number of operations and the scaling for solution on a fixed-point computing machine. *M. A. Hyman.*

Greenspan, Donald. Note on nine-point analogues of Laplace's equation. J. Franklin Inst. **264** (1957), 453–455.

Altman, M. On the convergence of Galerkin's approximate process in (B_0)-spaces. Bull. Acad. Polon. Sci. Cl. III. **5** (1957), 717–720, LXIII. (Russian summary)

Ergebnisse von N. I. Polsky [Ukrain. Mat. Ž. **7** (1955), 56–70; MR **17**, 64] über das Galerkinsche Verfahren in Banach-Räumen werden auf die von S. Mazur und W. Orlicz [Studia Math. **10** (1948), 184–208; MR **10**, 611] untersuchten B_0-Räume übertragen. Der B_0-Raum X besitze eine Basis $\{e_i\}$, so dass sich jedes $x \in X$ in der Form $x=\sum_{i=1}^{\infty} f_i(x)e_i$ darstellen lässt, wobei die f_i lineare stetige Funktionale mit $f_i(e_j)=\delta_{ij}$ bedeuten. A sei ein linearer vollstetiger Operator mit Definitionsbereich und Wertebereich in X. Zu lösen sei die Gleichung $x-hAx=y_0$ ($y_0 \in X$, h=const). Beim Galerkinschen Verfahren berechnet man Näherungen $x_n=\sum_{i=1}^{n} a_i^{(n)}e_i$, indem man die Konstanten $a_i^{(n)}$ so bestimmt, dass sie die linearen algebraischen Gleichungen $f_j(x_n-hAx_n-y_0)=0$ ($j=1, 2, \cdots, n$) erfüllen. Satz: Ist h ein regulärer Wert (d.h., kein Eigenwert) des Operators A, so existieren die Galerkinschen Näherungslösungen x_n, abgesehen höchstens von endlich vielen Indices n, und die Folge x_n konvergiert gegen eine Lösung der gegebenen Aufgabe.

J. Schröder (Hamburg).

Rubinstein, L. I. On the solution of Verigin's problem. Dokl. Akad. Nauk SSSR (N.S.) **113** (1957), 50–53. (Russian)

A special method is given for solving a partial differential equation boundary value problem that arises in oil industry, as was pointed out by N. N. Verigin [Izv. Akad. Nauk SSSR. Otd. Tehn. Nauk **1952**, 674–687]. This problem is reduced to the solution of a set of integral equations that can be solved by the method of successive approximations. The convergence of the process and the uniqueness of the solution are not proved, but the author states that this can be done by known methods.

H. P. Thielman (Ames, Iowa).

Dostupov, B. G. On a method of estimating the accuracy of compound integrators. Trudy Inst. Mašinoved. Sem. Točn. Mašinostro. Priborostr. **9** (1957), 52–66. (Russian)

Starting with a knowledge of errors in individual units of a differential analyzer, the author obtains a linearized error analysis of the behavior of a system of such units used in the solution of (a) a system of ordinary differential equations, and (b) ordinary differential equations of nth order. He used the method of the adjoint differential equations to determine, finally, the mean and standard deviation of the error of certain values at a specific time. A very complete example of the analysis of a specific complex differential analyzer problem is given. No formal limitations on the use of the linearized theory are given.

J. W. Carr, III (Ann Arbor, Mich.).

★ **Кармазина, Л. Н. [Karmazina, L. N.] Таблицы полиномов Якоби. [Tables of Jacobi polynomials.]** Izdat. Akad. Nauk SSSR, Moscow, 1954. 250 pp. 27.90 rubles.

The Jacobi polynomials, $G_n(p, q, x)=\sum_{i=0}^{n} A_i(p, q)x^i$, tabulated here are those which are orthogonal on the interval [0, 1] with respect to the weight function $x^{q-1}(1-x)^{p-q}$, normalized so that $A_n(p, q)=1$. They are given to 7D for $n=1(1)5$, $p=1.1(0.1)3.0$, $q=0.1(0.1)1.0$, and $x=0(.01)1.00$. The coefficients $A_i(p, q)$ ($i=0, \cdots, n-1$) are tabulated to 7S and the n zeros of G_n, to 7D for the same range of n, p, q. The special case $p=q=1$ gives the Legendre polynomials $P_n(x)$ which are tabulated to 7D for $x=0(.01).50$, $n=1(1)5$ together with their coefficients and zeros.

No differences are tabulated, and the introduction gives indications as to the accuracy of linear, quadratic, and cubic interpolation in the three variables x, p, q. Recurrence and asymptotic formulas are given to enable one to calculate the function for values of n, p, and q for which the function is not tabulated. The arrangement of the table is such that all the entries for a particular pair p, q are given on one page. *P. Rabinowitz.*

★ **Смирнов, А. Д. [Smirnov, A. D.] Таблицы функций Эйри и специальных вырожденных гипергеометрических функций для асимптотических решений дифференциальных уравнений второго порядка. [Tables of Airy functions and special degenerate hypergeometric functions for asymptotic solutions of differential equations of second order.]** Izdat. Akad. Nauk SSSR, Moscow, 1955. 264 pp. 28 rubles.

These tables were designed to enable one to compute asymptotic solutions of second order differential equations by the method of "the standard equation" given by Dorodnicyn [Uspehi Mat. Nauk (N.S.) **7** (1952), no. 6(52), 3–96; MR **14**, 876], although they are also of interest in themselves. There are three sets of tables as follows.

The first set gives the two independent solutions, $U_1(s)$ and $U_2(s)$, of the Airy equation $U''(s)+sU(s)=0$, with $U_1(0)=1$, $U_1'(0)=0$, $U_2(0)=0$, $U_2'(0)=1$, together with their first derivatives $U_1'(s)$ and $U_2'(s)$ and three auxiliary functions

$$V_{11}(s)=\int_0^s \Big(U_1^2(t)-\frac{3^{\frac{1}{3}}\Gamma^2(2/3)}{2\pi t^{\frac{1}{3}}}\Big)dt,$$

$$V_{12}(s)=\int_0^s \Big(U_1(t)U_2(t)-\frac{1}{2(3t)^{\frac{1}{3}}}\Big)dt,$$

$$V_{22}(s)=1+\int_0^s \Big)U_2^2(t)-\frac{\Gamma^2(1/3)}{2\pi 3^{\frac{1}{3}}t^{\frac{1}{3}}}\Big)dt.$$

These functions are used to find asymptotic solutions of the equation $y''+[\lambda^2 xr(x)+g(x)]y=0$. The functions $U_1(s)$, $U_2(s)$, $U_1'(s)$, $U_2'(s)$ and their backward second differences are tabulated to 5D for $s=-6.00(.01)10.00$, while the functions $V_{11}(s)$, $V_{12}(s)$, $V_{22}(s)$ are tabulated to 5D for $s=0(.01)10.00$.

The second set of tables is concerned with the solutions of the generalized Airy equation $U''(s)+s^\alpha U(s)=0$, satisfying the same initial conditions. $U_1(s, \alpha)$, $U_2(s, \alpha)$, $U_1'(s, \alpha)$, and $U_2'(s, \alpha)$ are tabulated to 4D for $s=0(.01)6.00$ and $\alpha=-3/4(1/4)2$, $\alpha=-2/3(1/3)5/3$, with $\alpha=0, 1$ excluded. First backward differences are tabulated only for $U_1(s, \alpha)$ and $U_2(s, \alpha)$. These functions are useful for finding asymptotic solutions of the equation $y''+[\lambda^2 x^\alpha r_\alpha(x)+g(x)]y=0$.

The third set consists of functions which enable one to find asymptotic solutions of the equation

$$y''+\Big(\lambda^2\frac{r(x)}{x}+\frac{p_0}{2}\Big(1-\frac{p_0}{2}\Big)\frac{1}{x^2}+\frac{g(x)}{x}\Big)y=0.$$

These special degenerate hypergeometric functions, $V_1(s, p_0)$ and $V_2(s, p_0)$, are two independent solutions of the equation

$$V''(s)+\left(\frac{p_0}{2}\left(1-\frac{p_0}{2}\right)\frac{1}{s^2}+\frac{1}{s}\right)V(s)=0.$$

V_1, V_2, V_1', and V_2' are tabulated to 4D for $s=0(.01)10.00$ and $p=0.1(.1)1.0$. First backward differences of V_1 and V_2 are given, as well as values of the first five zeros of V_1 and V_2. Asymptotic formulas for V_1 and V_2 are also given.

The introduction contains a full account of Dorodnicyn's method and gives an example to show how the tables are used in its application. A nomogram for finding $\frac{1}{2}u(1+u)x$, given x and $-u$, is appended for use with Newton's backward interpolation formula.

P. Rabinowitz (Rehovoth).

See also: Approximations, Orthogonal Functions: Tournarie. **Partial Differential Equations:** Zaikin and Kazarnovskiĭ; Glantz and Reissner. **Topological Vector Spaces:** Mirakov. **Computing Machines:** Gouarné; Kogbetliantz; Kreĭnes, Vaĭnšteĭn, and Aĭzenštat; Adler. **Statistics:** Czechowski, Fisz, Iwiński, Lange, Sadowski, and Zasepa; Farchi; Foster. **Mechanics of Particles and Systems:** Pavlenko. **Fluid Mechanics, Acoustics:** Miller; Payne.

Computing Machines

Leepin, P. Grundbegriffe der elektronischen Informationsverarbeitung. Mitt. Verein. Schweiz. Versich.-Math. **57** (1957), 224–230.

Rutishauser, H. Über die Vorbereitungsarbeit beim automatischen Rechnen. Mitt. Verein. Schweiz. Versich.-Math. **57** (1957), 247–257.

This is a tutorial paper for mathematicians in the procedure for solution of numerical problems on digital computers. In succession, the author deals with the following topics: the language of computing machines, execution of the arithmetic operations, a program for a particular formula, the structure of a problem, calculation with instructions, the path of control of the calculation, calculational error control, programming errors, and automatic programming. It is hoped that the author, who has led a great deal of development in this field, will soon make available to his readers a more elaborate expository account of automatic programming, which is an area that promises to revolutionize not only computational procedures, but also many parts of mathematics as well.

J. W. Carr, III (Ann Arbor, Mich.).

Gouarné, René. Calcul automatique des polynomes caractéristiques. C. R. Acad. Sci. Paris **245** (1957), 1114–1117.

This paper gives a flow diagram for computing the characteristic polynomial of an $n\times n$ matrix.

R. Hamming (Murray Hill, N.J.).

Kogbetliantz, E. G. Computation of e^N for $-\infty<N<+\infty$ using an electronic computer. IBM J. Res. Develop. **1** (1957), 110–115.

The authors studies rational and polynomial approximations to the exponential function, for binary and decimal electronic computers, operating in the fixed and in the floating point mode. For each of these combinations, methods using fewest operations and thus least time are presented.

W. F. Freiberger (Providence, R.I.).

Tocher, K. D. The classification and design of operation codes for automatic computers. Proc. Inst. Elec. Engrs. B. **103** (1956), supplement no. 1, 125–133.

The author studies the respective requirements which programmer and engineer might advance for the logical design of an automatic computer. The study is made in terms of the type of storage used and the mode of access to it which is provided.

The system studied in detail is the rather unusual one in which separate storage systems are used for numbers and instructions. Certain advantages are claimed for this system, although the recent advent of very large high-speed stores — not available at the time the paper was written — must invalidate some of the arguments.

W. F. Freiberger (Providence, R.I.).

Hill, G. W.; and Pearcey, T. Programme design for the C.S.I.R.O. Mark I computer. IV. Automatic programming by simple compiler techniques. Austral. J. Phys. **10** (1957), 137–161.

The authors discuss automatic programming techniques developed for the C.S.I.R.O. computer. These are both compiler programmes and interpreter routines.

W. F. Freiberger (Providence, R.I.).

Fröberg, Carl-Erik; and Wahlström, Gunnar. The SMIL computer in Lund. Lunds Univ. Årsskr. N. F. Avd. 2. **53**=Kungl. Fysiogr. Sällsk. i Lund Handl. N. F. **68** (1957), no. 4, 38 pp. (Swedish. English summary)

Shapiro, Elmer B. A command code translator for the ORDVAC. Ordnance Computer Research Report, Ballistic Research Laboratories, Aberdeen Proving Ground, Md. vol. 4 (1957), no. 3, pp. 7–10. (Government Agencies, their contractors and others cooperating in Government research may obtain reports directly from the Ballistic Research Laboratories. All others may purchase photographic copies from the Office of Technical Services, Department of Commerce, Washington 25, D. C.)

Schäfer, H.-W. Praktische Herstellung und Prüfung eines Programms für elektronische Rechenautomaten. Mitt. Verein. Schweiz. Versich.-Math. **57** (1957), 258–259.

Schläpfer, J. W. Erfahrungen bei der Einführung von grossen elektronischen Data Processing Maschinen. Mitt. Verein. Schweiz. Versich.-Math. **57** (1957), 281–285.

Obuvalin, M. I.; and Èterman, I. I. On the solution of boundary value problems using devices intended for the solution of the problem of Cauchy. Inžen. Sb. **23** (1956), 203–213. (Russian)

This paper describes the use of an analog computer, the MPT-9 in Moscow, to obtain solutions to the boundary value problems of ordinary differential equations with coefficients functions of the independent variable.

The first method consists of separate solutions of the corresponding initial value problem on the analog computer for cases where all of the initial values are non-zero except for only the function or one of its derivatives. These solutions are used to solve by hand for the required initial values needed to obtain the desired solution at the far end. The analog computer is then used to get an approximation to the desired solution, and the initial values are then varied to improve performance.

The second method described is to use the analog computer to solve the analogous finite difference problem, from this to obtain approximations to the initial conditions by hand calculation and then to repeat the analog solution using these initial values. No error analysis is included, and the most complex example is a fourth order equation. Tables of formulas for the first four derivatives in terms of finite differences are included.

J. W. Carr, III (Ann Arbor, Mich.).

Kreines, M. A.; Vainštein, I. A.; and Aĭzenštat, N. D. About a device for construction of approximate nomograms. Dokl. Akad. Nauk SSSR (N.S.) **110** (1956), 922–925. (Russian)

The authors show that, using a device consisting of a frame, thin-wire needles, and slide-blocks, they can determine a nomogram for a function which will approximate a certain class of functions $z=f(x, y)$, continuous in each variable and monotonically increasing in each variable when the other is held constant. Unfortunately, the conditions of the two theorems proved relate only to tests able to be made by means of the frame, needles, and slide-block arrangement, and there is no useful test given for those persons not in possession of that mechanism. The paper may be of use to persons wishing to construct such nomograms in that its technique might be translatable into a trial and error process using a digital computer. A transformation of the (x, y) plane to a (u, v) plane, preserving the linear nature of the lines x=const. and y=const. and then the existence of a third family of lines representing z=constant is searched for by a trial and error process. *J. W. Carr, III* (Ann Arbor, Mich.).

Kapica, S. P. Mechanical calculation of harmonically conjugate functions. Vyčisl. Mat. **1** (1957), 167–169. (Russian)

The author presents a mechanism for an approximate calculation of

$$w(x)=\text{prin.pt.}\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{u(\xi)}{\xi-x}\,d\xi,$$

thus enabling him "to solve one of the basic problems of the theory of functions of a complex variable: i.e., to find the value of a function of a complex variable at any regular point of the plane, knowing only the real (or imaginary) part on a straight line."

The mechanism is a wheel and disc integrator, attached to a sliding coulisse, that actually obtains the integral

$$n=\frac{1}{\pi}\int_a^b\frac{u(\xi)-u(x)}{\xi-x}\,d\xi$$

approximately equivalent to the first integral above, as an angle of rotation of a small wheel on a larger disc.

This device, while interesting geometrically, is probably not of as much use today as it would have been ten to fifteen years ago when the mechanical differential analyzers were the only practical method of computing such an integral. *J. W. Carr, III* (Ann Arbor, Mich.).

★ **Adler, H. Ein elektrisches Gerät zur Lösung von Polynomgleichungen. Aktuelle Probleme der Rechentechnik.** Bericht über das Internationale Mathematiker-Kolloquium, Dresden, 22. bis 27. November 1955, pp. 93–97. VEB Deutscher Verlag der Wissenschaften, Berlin, 1957.

After a brief review of some existing instruments for finding the real and complex roots of a polynomial $f(z)$ the author describes a rather inexpensive electronic analogue computer of the Technical University of Dresden. The powers $z^n=r^n(\cos n\theta+i\sin n\theta)$ are generated for a discrete set of values $r=r_k$, $\theta=\theta_k$ by means of potentiometers and transformers; in the same way the combination of the powers is obtained; $f(z)$ is shown on the screen of a cathode ray tube. The solution of $f(z)=0$ is by trial and error. *H. Bückner* (Schenectady, N.Y.).

See also: Theory of Algebraic Numbers: Cohn and Gorn. **Numerical Methods:** Miroux; Dostupov. **Economics, Management Science:** Goode. **Control Systems:** Doll and Stout; Gendreu.

PROBABILITY

★ **Gnedenko, B. W. Lehrbuch der Wahrscheinlichkeitsrechnung.** Mathematische Lehrbücher und Monographien, herausgegeben von der Deutschen Akademie der Wissenschaften zu Berlin, Forschungsinstitut für Mathematik. I. Abteilung, Mathematische Lehrbücher. Bd. IX. Akademie-Verlag, Berlin, 1957. xi+387 pp. DM 29.50.

A German translation, by W. Richter under the technical editorship of H. J. Rossberg, of a text prepared for the second Russian edition (1954) of the original Russian first edition reviewed in MR **13**, 565. The most important change made in the second edition consists in the addition of exercises for the first nine chapters.

★ **von Mises, Richard. Probability, statistics and truth. Second revised English edition, prepared by Hilda Geiringer.** The Macmillan Company, New York, 1957. xiv+244 pp. $5.00.

Translation of the third German edition, reviewed in MR **10**, 837, with minor deviations; amongst these, the references have been brought up to date, the notes modified and a subject index has been added.

Gnedenko, B. V. On certain problems of the theory of probability. Ukrain. Mat. Ž. **9** (1957), 377–388. (Russian. English summary)

This paper presents a review of urgent problems in the following domains of the theory of probability: 1) limit theorems for sums of independent random variables; 2) probability methods in the theory of numbers; 3) application of functional analysis in the theory of probability; 4) the theory of information; 5) problems of mass service; 6) application to computation mathematics; 7) theory of games; 8) theory of decision functions and other problems of mathematical statistics.

Author's summary.

Cramér, Harald. Ein Satz über geordnete Mengen von Wahrscheinlichkeitsverteilungen. Teor. Veroyatnost. i Primenen. **1** (1956), 19–24. (Russian summary)

Under simple assumptions, the author proves that if an ordering $\leqq$ on a set A can be extended to the class X_2 of probability measures on A which are supported by two points, then there is a function f on A such that, for x, y in X_2, $x>y$ if and only if $\int f(a)dx(a)<\int f(a)dy(a)$. In previous

work in this area [e.g., von Neumann and Morgenstern, Theory of games and economic behaviour, 2nd ed., Princeton, 1947; MR 9, 50; Herstein and Milnor, Econometrica 21 (1953), 291–297; MR 15, 812], X_2 was usually replaced by the class of all probability measures with finite support, in both hypothesis and conclusion.

J. Kiefer (Ithaca, N.Y.).

Hanš, Otto. Reduzierende zufällige Transformationen. Czechoslovak Math. J. 7(82) (1957), 154–158. (Russian summary)

The following is the main result of the paper. Consider an abstract space Ω with a σ-algebra S, and also a complete, separable metric space X. Let T be a reducing measurable transformation from $\Omega \times X$ onto X, choose a generalized stochastic variable $g_1(\omega)$ with values in X, and form $g_{n+1}(\omega)=T[\omega, g_n(\omega)]$. Then the sequence $g_1(\omega)$, $g_2(\omega)$, $\cdots$ converges to a generalized stochastic variable, which is the fixed point associated with T. This theorem is applied to the study of integral equations of Fredholm type.

U. Grenander (Providence, R.I.).

Heaps, H. S. The reflection coefficient of a surface of Rayleigh distributed impedance. Quart. Appl. Math. 15 (1957), 291–297.

The author obtains an expression for the probability distribution of the amplitude k of $(z-c_2)^2(z+c_1)^{-2}$, where c_1, c_2 are real constants and $z=c+z_1$ is a complex-valued random variable. c is a complex constant while z_1 has a normally distributed amplitude and a uniformly distributed phase. The case $c_1=c_2$ arises when one considers the reflection of a plane wave of sound from a plane surface. k is then the power reflection coefficient and z the acoustic impedance ratio.

M. Rosenblatt (Bloomington, Ind.).

Linnik, Yu. V. On factorizing the composition of a Gaussian and a Poissonian law. Teor. Veroyatnost. i Primenen. 2 (1957), 34–59. (Russian. English summary)

Author's summary: The paper contains a detailed proof for the following theorem: the composition of a Gaussian and a Poissonian law can be factorized only into similar compositions. More precisely, let $X=X_1+X_2$ where X_1 is a Gaussian component and X_2 a Poissonian component independent of X_1. If there is some other factorization $X=Y_1+Y_2$, Y_1 being independent of Y_2, then $Y_1=Y_{11}+Y_{12}$, $Y_2=Y_{21}+Y_{22}$, where Y_{11}, Y_{21} are Gaussian and Y_{12}, Y_{22} Poissonian, all mutually independent, and $D(Y_{11})+D(Y_{21})=D(X_1)$, $D(Y_{12})+D(Y_{22})=D(X_2)$. Thus, H. Cramér's theorem on factorizing the normal law and D. A. Raykov's theorem on factorizing the Poissonian law are special cases of this theorem.

G. E. Noether (Boston, Mass.).

Rizzi, Alfredo. Osservazioni sulle classi di Fréchet delle funzioni di ripartizione a più variabili. Boll. Un. Mat. Ital. (3) 12 (1957), 269–277.

The paper considers Fréchet's classes of real functions in R_2, R_3 and R_4, monotonic, with total variation equal to 1.

O. Onicescu (Bucarest).

Sukhatme, Balkrishna V. Joint asymptotic distribution of the median and a U statistic. J. Roy. Statist. Soc. Ser. B. 19 (1957), 144–148.

Let $\tilde{X}$ be the sample median of $2n+1$ independent random variables $X_1, \cdots, X_{2n+1}$ having a common probability density with median θ, and let $U=(2n+1)^{-1}\sum \psi(X_k-\theta)$, where ψ is real-valued and bounded. It is shown by means of characteristic functions that, subject to mild regularity conditions, the joint distribution of $\tilde{X}$ and U is asymptotically normal as $n\to\infty$. Using theorems of the reviewer [Ann. Math. Statist. 19 (1948), 293–325; MR 10, 134] and E. L. Lehmann [ibid. 22 (1951), 165–179; MR 12, 726], this result is then extended to the case where U is proportional to the symmetric sum

$$\sum \psi(X_{\alpha_1}-\theta, \cdots, X_{\alpha_n}-\theta)$$

(where $\alpha_i \neq \alpha_j$ if $i \neq j$) and to an analogous case involving two samples.

W. Hoeffding (Chapel, Hill, N.C.).

Esseen, C. G. A moment inequality with an application to the central limit theorem. Skand. Aktuarietidskr. 39 (1956), 160–170 (1957).

Let $X_1, \cdots, X_n, \cdots$ be independent random variables with the same distribution function (d.f.) $F(x)$, $E(X_n)=0$, $E(X_n^2)=\sigma^2>0$, $E(X_n^3)=\alpha_3$, $E(|X_n|^3)=\beta_3<\infty$. Let $\rho(F_n,\Phi) = \sup_x |F_n(x)-\Phi(x)|$, where $F_n(x)$ denotes the d.f. of $(X_1+\cdots+X_n)\sigma^{-1}n^{-\frac{1}{2}}$ and $\Phi(x)=(2\pi)^{-\frac{1}{2}}\int_{-\infty}^{x}\exp(-t^2/2)dt$. It is known [A. C. Berry, Trans. Amer. Math. Soc. 49 (1941), 122–136; MR 2, 228; C. G. Esseen, Acta Math. 77 (1945), 1–125; MR 7, 312] that $n^{\frac{1}{2}}\rho(F_n,\Phi)\leq C\beta_3/\sigma^3$, where C is a numerical constant. From the author's earlier work it follows that

$$\lim_{n\to\infty} n^{\frac{1}{2}}\rho(F_n,\Phi)=(2\pi)^{-\frac{1}{2}}(3h\sigma^2+|\alpha_3|)/(6\sigma^3),$$

where h is the maximum span of F if F is a lattice distribution and $h=0$ otherwise. An inequality is proved which implies that $3h\sigma^2+|\alpha_3|\leq(10^{\frac{1}{2}}+3)\beta_3$, where equality can be attained. Hence $C\geq(2\pi)^{-\frac{1}{2}}(10^{\frac{1}{2}}+3)/6$.

W. Hoeffding (Chapel Hill, N.C.).

Dugué, Daniel. Sur le comportement asymptotique des suites de variable aléatoires. C. R. Acad. Sci. Paris 244 (1957), 2885–2886.

L'auteur démontre le théorème suivant: Si $X_1, X_2, \cdots$ et $Y_1, Y_2, \cdots$ sont deux suites aléatoires, les X étant indépendants des Y, et si X_n+Y_n converge en probabilité vers une limite aléatoire, il existe deux suites de nombres certains $a_1, a_2, \cdots$ et $b_1, b_2, \cdots$, tels que les suites $X_1-a_1, X_2-a_2, \cdots$ et $Y_1-b_1, Y_2-b_2, \cdots$ tendent en probabilité vers une limite aléatoire.

J. Wolfowitz.

Takács, L. On some probability problems concerning the theory of counters. Acta Math. Acad. Sci. Hungar. 8 (1957), 127–138.

This paper continues previous research of the author on recurrent events [Teor. Veroyatnost. i Primenen. 1 (1956), 90–102; MR 18, 831]. The terminology and notation of the previous paper will be used here. Let $t_1, \cdots, t_n$ be the instants at which events occur, and $t_1', \cdots, t_n'$ the instants at which an event occurs while no impulse is in progress. Denote by ν_t the number of events $\{t_n'\}$ occurring in $(0, t]$. It is shown that the limit distribution of ν_t is normal. The parameters of the limit distribution of ν_t can be obtained by means of integral equations from the distribution of t_1 and of t_1'. The calculations are carried through in the case that χ_n is constant and in the case that it has an exponential distribution. Denoting by $P(t)$ the probability that at time t no impulse is in progress, the author further shows that under certain conditions $\lim_{t\to\infty} P(t)$ exists.

The author then considers a system of n counters where the distributions of the arrivals and the duration

of the impulses may be different for different counters. The system is at time t in the state E_k if at time t exactly k impulses are in progress. A transition from the state E_{k-1} to the state E_k is called a k-fold chance coincidence. The author determines the asymptotic density of k-fold chance coincidences. *H. B. Mann* (Columbus, Ohio).

Winkelbauer, Karel. On the theory of generalized random processes. Czechoslovak Math. J. **6(81)** (1956), 517–521. (Russian. French summary)

Let X be the class of functions on k-space which are complex-valued, infinitely differentiable, and have compact supports. Let $\tilde{X}$ be the space of all Schwartz distributions on X. Let F be the space of complex-valued functions on X. Let $\mathfrak{F}$ be the minimal σ-algebra of subsets of F containing those of the form

$$\{f: \operatorname{Re} f(\phi)<a, \operatorname{Im} f(\phi)<b\}$$

where $\phi \in X$. Let μ be a probability measure on $\mathfrak{F}$. Then it is proved that the outer μ measure of $\tilde{X}$ is 1 if and only if the following two conditions are satisfied. I. If $\phi_i \in X$, $f(\phi_1)+f(\phi_2)=f(\phi_1+\phi_2)$ for almost all f in F (μ measure). II. (Let $k=1$; the condition in the general case is the natural reformulation.) If $\varepsilon>0$, and if m is a natural number, there is an integer $N(\varepsilon, m)$ such that, if $r \geqq N(\varepsilon, m)$, and if S is a finite subset of elements in X which vanish outside $[-m, m]$, then

$$\mu\Big(\bigcap_{\phi \in S} \{f: |f(\phi)| \leqq r(2m)^r \sup |\phi^{(n)}|\}\Big) \geqq 1-\varepsilon.$$

If these conditions are satisfied, the triple $(F, \mathfrak{F}, \mu)$ defines a measure on the space $\tilde{X}$. [For other approaches to such measures, see I. M. Gelfand, Dokl. Akad. Nauk SSSR (N.S.) **100** (1955), 853–856; MR **16**, 938; and K. Itô, Mem. Coll. Sci. Univ. Kyoto. Ser. A. Math. **28** (1954), 209–223; MR **16**, 378.] *J. L. Doob* (Urbana, Ill.).

Breny, H. Recherches sur la théorie statistique des faisceaux de fibres. Mém. Soc. Roy. Sci. Liège (4) **17** (1957), no. 2, 131 pp.

This is an interesting application of the theory of stochastic processes or time series analysis to textile fibre bundles. It is assumed that the bundle is very long. Then the number of fibres in the bundle follows a binomial distribution. The number of fibres between certain limits follows a Poisson distribution; samples of the number of fibres are also distributed according to a Poisson distribution.

The joint probability distribution of the number of fibres in a bundle at various points is established with the help of the method of generating functions. It converges towards a Poisson distribution which, in general, has a variable mean.

Indices of irregularity are established with the help of an operator $\operatorname{Erg}_t = \lim (a+b)^{-1} \int_{-a}^{b} f(t)dt$ $(a, b \to \infty)$, where $f(t)$ is a local index of irregularity. Point, longitudinal, local, mean and global indices of irregularity are considered, as are indices based upon moving averages with exponential weights. Some of the indices have stationary properties, even if the underlying stochastic process is not stationary to the second order.

Ergodic properties are investigated for the number of fibres in a given location, length of fibres etc. Ergodic variance, ergodic autocorrelation, ergodic indices of irregularity are considered. Spectral analysis is carried out with the help of the integrated spectrum.

G. Tintner (Ames, Iowa).

Zítek, František. On estimating transition probabilities. Apl. Mat. **2** (1957), 251–257. (Czech. Russian and English summaries)

Methods of estimation of transition probabilities in a homogeneous two-state Markov chain are considered. Lange's solution is valuable only if n_i (the number of transitions from the state i to any other) is fixed. Advantageous properties of Haldane's method for estimates are given. Confidence intervals for transition probabilities are constructed using confidence intervals for the parameter of the binomial distribution. Different methods of estimation are illustrated by a numerical example.

J. Janko (Praque).

Jacobs, Konrad. Fastperiodische diskrete Markoffsche Prozesse von endlicher Dimension. Abh. Math. Sem. Univ. Hamburg **21** (1957), 194–246.

This paper deals with discrete parameter Markov chains with a finite number of states. Such a process is viewed as the successive partial products of a given sequence of stochastic matrices, identical (stationary) or not. The terminology and methodology of linear space are employed in full. E.g., the Markov theorem is proved by Minkowski's theorem on extremal points of a convex set. If the sequence of matrices is almost periodic in norm, it is shown that their partial products are asymptotically almost periodic. Similar investigation was made in less sophisticated setting by Fréchet [Rev. Sci. **79** (1941), 341–354; MR **7**, 127]. *K. L. Chung* (Syracuse, N.Y.).

Neveu, Jacques. Sur le comportement asymptotique des chaînes de Markov. C. R. Acad. Sci. Paris **245** (1957), 493–496.

Using the theory of Riesz spaces, the asymptotic behavior of Markov chains with a finite number of states is studied. In the temporarily homogeneous case the results give the classical decomposition into cyclic classes.

K. L. Chung (Syracuse, N.Y.).

Kawamura, Tomoo. On Markov chains. I. Rep. Statist. Appl. Res. Un. Jap. Sci. Engrs. **4** (1957), 111–124.

In a temporally homogeneous discrete parameter Markov chain (with a denumerable number of states) the time of one-step transition from j to k is supposed to be t_{jk} (instead of 1), where $0<t_{jk}<K$. The known results on mean recurrence time are then generalized to this case.

K. L. Chung (Syracuse, N.Y.).

Ramakrishnan, Alladi. Ergodic properties of some simple stochastic processes. Z. Angew. Math. Mech. **37** (1957), 336–344. (German, French and Russian summaries)

The author's aim is to find a process for which the ergodic theorem is readily demonstrated and appreciated. He considers a time-homogeneous Markov process with a finite number of states, for which the transition intensities exist, and for which any pair of states can be linked by at least one chain of non-zero intensities. He then shows that the proportion of the time interval $(0, t)$ during which a particular state is occupied converges in probability to the corresponding occupation probability as t tends to infinity. *P. Whittle* (Wellington).

Zítek, František. On a theorem of Korolyuk. Czechoslovak Math. J. **7(82)** (1957), 318–319. (Russian. English summary)

In a stationary stream of events, let μ be the expected number of events per unit time, and let λ be the limit, as

$t\to 0$, of (probability that at least one event occurs in time t)/t. According to a theorem of Korolyuk [Hinčin, Trudy Mat. Inst. Steklov. **49** (1955); MR **17**, 276], if (A) only one event can occur at a time, then $\mu=\lambda$. The author proves that, if $\mu<\infty$, the condition (A) is both necessary and sufficient that $\mu=\lambda$. If $\mu=\infty$, then (A) implies that $\mu=\lambda$, but there are processes not satisfying (A) for which $\mu=\lambda=\infty$. *J. L. Doob* (Urbana, Ill.).

★ **Cramér, Harald. Collective risk theory: A survey of the theory from the point of view of the theory of stochastic processes.** Reprinted from the Jubilee Volume of Försäkringsaktiebolaget Skandia. Skandia Insurance Company, Stockholm, 1955. 92 pp.

Continuing an earlier survey [Försäkringsaktiebolaget Skandia, 1855–1930, v. 2, Stockholm, 1930, pp. 7–84], the author surveys the progress of collective risk theory since 1930. The mathematical problem is formulated as follows. Events occur in accordance with a Poisson process. If $n(t)$ have occurred in the interval $[0, t]$, $Y(t)$ is the sum of $n(t)$ independent random variables with the common distribution function P. Let $p_1=E\{Y(1)\}$, and define $X(t)=p_1t-Y(t)$. The problem is to analyze the asymptotic character of the $Y(t)$ process for large t and to find the probability that its sample function graphs ever cross certain lines. The first three chapters detail the background of the problem. The fourth is in greater part devoted to Esscher's work [Skand. Aktuarietidskr. **15** (1932), 175–195] on the first terms of an asymptotic series for Prob. $\{X(t)\geqq cte^{\beta t}\}$, where β is a certain positive constant and c is any positive constant. Chapter 5 discusses the ruin problem. If λ is a strictly positive constant, and if u is a positive constant, $\psi(u, T)$ is defined as Prob.$\{\inf_{0<t<T} X(t)+u+\lambda t<0\}$. Among other results, an integral equation for ψ is found, and is solved to yield explicit expressions for $\psi(u, T)$ and $\psi(u, \infty)$. The principal results in this ruin problem are due to Arfwedson, Cramér, Lundberg, Saxén, Segerdahl, and Täcklind. Chapter 6 contains a brief discussion of the generalizations of the previous work, in which now P may depend on the time an event occurs, and in which λ is allowed to vary ("variable safety loading"). *J. L. Doob.*

Bellman, Richard; and Kalaba, Robert. Random walk, scattering, and invariant imbedding. I. One-dimensional discrete case. Proc. Nat. Acad. Sci. U.S.A. **43** (1957), 930–933.

★ **Ulam, S. M. Infinite models in physics.** Applied probability. Proceedings of Symposia in Applied Mathematics, Vol. VII, pp. 87–95. McGraw-Hill Book Co., New York-Toronto-London, for the American Mathematical Society, Providence, R. I., 1957. $5.00.

★ **Münch, Guido. Stochastic processes of astronomical interest.** Applied probability. Proceedings of Symposia in Applied Mathematics, Vol. VII, pp. 51–66. McGraw-Hill Book Co., New York-Toronto-London, for the American Mathematical Society, Providence, R. I., 1957. $5.00.

In mathematical terms, the problem studied is the following. Consider the stochastic integral

$$U(t)=\int_0^t q^{N(s)}ds,$$

where q is a positive constant less than one and $N(s)$ is the integer-valued process associated with the Poisson point process. To study the distribution of this integral, the author uses the Markovian nature of the problem to get a functional equation for the distribution function. This equation can be reduced to a partial differential equation which is solved explicitly. The author also considers the more general case when

$$U_\varphi(t)=\int_0^t \prod_{\nu=1}^{N(s)} q_\nu ds,$$

where the q_ν are independent stochastic variables with the given probability law $\varphi(q)$. While the corresponding functional equation has not been solved, the moments of the distribution are determined, as is also the limiting distribution for $t\to\infty$. *U. Grenander.*

Vorob'ev, A. P. Some investigations on the rolling of a ship by the methods of the theory of probability. Leningrad. Gos. Univ. Uč. Zap. **217**. Ser. Mat. Nauk **31** (1957), 3–16. (Russian)

The problem of the rolling of a ship leads to the problem of the integration of the stochastic differential equation

$$J\frac{d^2\theta}{dt^2}+a\frac{d\theta}{dt}+b\theta=r(t),$$

where $r(t)$ is a random function. The stationary case is especially interesting, as well as that of the process with uncorrelated increases. If we assume that the initial values are random ones, the solutions may not generally be stationary random functions. The author considers cases where the solution is stationary as well. Assuming that $r(t)$ is given by a series $\sum A_k \sin(\lambda_k t+\alpha_k)$, where A_k and α_k are random variables, the author points out, among other things, that the phenomenon of resonance is maintained if the λ_k tend to the values of the natural frequencies of the vessel. *O. Onicescu* (Bucarest).

Barrer, D. Y. Queuing with impatient customers and indifferent clerks. Operations Res. **5** (1957), 644–649.

The author presents a queuing problem in which the arrival and service rates are both Poisson, and each customer limits his waiting time to a fixed period τ_0 after which he leaves the system and is considered lost. Customers are chosen for service at random from the queue. They are considered to be still on line until their service is completed. Hence, a customer will be lost if he has waited τ_0 even if he is designated for service. The problem is formulated for the transient case, but a solution is obtained only under the assumption of statistical equilibrium. In this case, if λ is the arrival rate, μ the service rate, and $\rho=\lambda/\mu$, then the probability of a customer being lost is

$$Q=\frac{1-(1-\rho)\sum_{n=1}^{\infty}\rho^{n-1}\prod_{k=1}^{n}[1-e^{-\mu\tau_0/k}]}{1+\sum_{n=1}^{\infty}\rho^{n}\prod_{k=1}^{n}[1-e^{-\mu\tau_0/k}]}$$

H. M. Gurk (Princeton, N.J.).

Barrer, D. Y. Queuing with impatient customers and ordered service. Operations Res. **5** (1957), 650–656.

The author presents two waiting line problems which are natural sequels to that considered in the paper reviewed above. In both problems, service and arrivals are Poisson, and customers are chosen from the queue on a first-come, first-served basis. Both problems are solved for the case of statistical equilibrium. In the first problem, there are s servers ($s=1, 2, \cdots$), and each customer will

wait no more than τ_0 until he is acquired for service. Once acquired he will be serviced no matter how much longer it takes. If not acquired before τ_0, the customer is considered lost. With λ and μ the arrival and individual service rates respectively, and $\rho=\lambda/\mu$, the probability that a customer is lost is

$$Q=\frac{(\rho-s)\rho^s e^{\mu\tau_0(\rho-s)}}{(\rho-s)s!\sum_{k=0}^{s}(\rho^k/k!)+\rho^{s+1}(e^{\mu\tau_0(\rho-s)}-1)}.$$

The second problem is the single server case in which the waiting time limit extends until the completion of service. With the same notation as above

$$Q=\frac{(1-\rho)e^{\mu\tau_0(\rho-1)}}{1-\rho e^{\mu\tau_0(\rho-1)}}.$$

H. M. Gurk (Princeton, N.J.).

See also: Measure, Integration: Kampé de Fériet. **Biology and Sociology:** Rapoport. **Information and Communication Theory:** Middleton and Van Meter.

STATISTICS

Barra, J. R. Sir R. A. Fisher's "Statistical methods and scientific inference". Publ. Inst. Statist. Univ. Paris **6** (1957), 3–10.

Expository paper. *H. Rubin* (Eugene, Ore.).

Thionet, P. Sur les agrégats de distributions statistiques d'une même famille. Publ. Inst. Statist. Univ. Paris **6**, (1957), 27–52.

The author investigates the "aggregate" distribution of economic wealth that is composed of two "base" distributions. Using the notion of the curve of concentration [see Gini, Metron **9** (1932), 3–76], he states the following result: Given an arbitrary family of distributions of a variable x having an arbitrary number of parameters for which the aggregate of these base distributions belongs to the same family, then it is necessary that the arithmetic means of the base distributions be equal to one another. The author notes that this condition is not sufficient [see also Fréchet, Rev. Inst. Internat. Statist. **7** (1939), 32–38].

H. P. Edmundson (Santa Monica, Calif.).

Somermeijer, W. H. Substitute variables in correlation analysis. Statistica Neerlandica **11** (1957), 153–160. (Dutch. English summary)

Warning against a shortcut sometimes practised, the author stresses that if two variables x, x^* are strongly intercorrelated this does not guarantee that x can replace x^* in correlation and regression analysis. Numerical illustrations. *H. Wold* (Uppsala).

Landenna, Giampiero. Osservazioni sulla connessione. Statistica, Bologna **17** (1957), 351–392.

The author studies various measures of association between the components of a two-dimensional random vector. The relations between these measures, as well as the connection with the "index of dissimilarity" investigated in an earlier paper [Statistica, Bologna **16** (1956), 21–57; MR **18**, 242] are also discussed. The last section treats the corresponding sample characteristics for a bivariate population with a finite number of possible values. *E. Lukacs* (Washington, D.C.).

Stevens, W. L. Shorter intervals for the parameter of the binomial and Poisson distributions. Biometrika **44** (1957), 436–440.

★ **Czechowski, T.; Fisz, M.; Iwiński, T.; Lange, O.; Sadowski, W.; i Zasępa, R. Tablice statystyczne. [Statistical tables.] Edited by Wiesław Sadowski.** Państwowe Wydawnictwo Naukowe, Warsaw, 1957. 158 pp. 32 zł.

This collection contains the usual "classical" tables and, in addition, a number of useful tabulations of recent origin. Among the former are tables of the normal, binomial, Poisson, Chi-square, t, z, and F distributions; a page of tables dealing with the sample correlation coefficient; random numbers; assorted auxiliary tables such as binomial coefficients, squares, cubes, roots, logarithms, etc. Among the newer tables are those dealing with distribution-free techniques such as run-tests, sign-test, Kolmogorov's statistic D_n, Smirnov's $D_{m,n}$, and two entirely new tables of "golden" and "iron" numbers due to H. Steinhaus. The tables are selected and edited with great care, and the explanatory text is excellent.

Z. W. Birnbaum (Seattle, Wash.).

Farchi, Vittorio. Le curve degeneri del sistema algebrico delle parabole dei minimi quadrati. Rend. Mat. e Appl. (5) **15** (1956), 291–314 (1957).

{Editor's note. The article reviewed here was listed by title in MR **18**, 822.}

Data una variabile casuale doppia a variabilità superficiale non nulla che ammette finiti tutti i momenti fino al quarto ordine, l'A. studia il sistema della parabole di secondo grado dei m.q. (minimi quadrati) rispetto a tutte le direzioni.

L'A. dimostra che, in queste ipotesi, vi sono al più tre direzioni reali, dette direzioni caratteristiche, rispetto alle quali la relativa parabola dei m.q. si spezza nella retta all'infinito e in una retta al finito (che sarà la relativa retta dei m.q.). L'A. dimostra inoltre che, nelle ipotesi poste, condizione necessaria e sufficiente, perchè il sistema delle parabole dei m.q. si riduca al fascio delle rette dei m.q., è che siano nulli tutti i momenti del terzo ordine rispetto alla media; vedi ad esempio la variabile casuale doppia normale. *G. Pompilj* (Roma).

Ihm, P. Berechnung von Integralen der n-dimensionalen Student-Verteilung mittels Stieltjesintegralpapier. Mitteilungsbl. Math. Statist. **9** (1957), 143–146.

The n-dimensional Student distribution in question is that with ν degrees of freedom and density function proportional to $[1+(t_1^2+\cdots+t_n^2)/\nu]^{-(\nu+n)/2}$; the integrals are of this function over given ranges of the variables t_1 to t_n. Integration is accomplished by using the following integral representation of the density:

$$\int_0^\infty z^{n/2}\exp-\frac{z}{2\nu}(t_1^2+\cdots+t_n^2)z^{(\nu/2)-1}\exp-\frac{z}{2}dz.$$

The integrals with respect to t_1 to t_n may be made first; for fixed z, they may be read from tables of the n-dimensional Gaussian function. The final integration is made numerically (or by use of planimeter) with the integrand written in Stieltjes form: $F_n(z)dG_\nu(z)$ and

$$G_\nu(z)=\int_0^z z^{(\nu/2)-1}e^{-z/2}dz.$$

(The operation of inverting the equation $x=G_\nu(z)$ leads to "Stieltjesintegralpapier".) *J. Riordan.*

Finney, D. J. The consequences of selection for a variate subject to errors of measurement. Rev. Inst. Internat. Statist. **24** (1956), 1–10.

A certain percentage P at the upper tail of measured variates (y) is selected for future research. It is desired to study the distribution of the true values (x) of the variate in the selected portion where $y=x+w$, w assumed $N(0, \varepsilon^2)$ and independent of x. Formulas are given for the cumulants of X and Y (the variates in the selected portion of the populations) in terms of ε^2 and the cumulants of x. Special results are given for x normally distributed. Tables of the constants appearing in these formulas are included for $P=0.025(0.025)0.10(0.05)0.95$ and selected smaller values of P. *R. L. Anderson.*

Fox, Charles. Some applications of Mellin transforms to the theory of bivariate statistical distributions. Proc. Cambridge Philos. Soc. **53** (1957), 620–628.

It is shown how two dimensional Mellin transforms can be used to find the p.d.f.'s of bivariate random variables which are the products or ratios of other bivariate random variables. The one dimensional case has been treated by the reviewer [Ann. Math. Statist. **19** (1948), 370–379; MR **10**, 552]. *Benjamin Epstein* (Stanford, Calif.).

Briggs, F. E. A. On problems of estimation in Leontief models. Econometrica **25** (1957), 444–455.

Least squares vs. maximum likelihood estimation are compared in the input-output model $Y=By+E$, where Y is the transaction matrix, B are technical coefficients, and E are random residuals. Six cases are explored: (a) The model is open or closed; (b) In closed models Say's law may or may not hold; (c) The vector y is or is not random. Three types of results: (1) In two cases the specification does not admit maximum likelihood estimation of B; (2) In two cases the two estimation methods are equivalent; (3) In two cases the two methods differ systematically, and an empirical illustration shows that the difference may be substantial. Briggs comments that the dualism in (3) disappears if the model is made dynamic instead of static. {It is unclear to the reviewer how the model should be used for applied work in situations (1) and (3).} *H. Wold* (Uppsala).

Theil, H.; and van Yzeren, J. On the efficiency of Wald's method of fitting straight lines. Rev. Inst. Internat. Statist. **24** (1956), 17–26.

Wald [Ann. Math. Statist. **11** (1940), 285–300; MR **2**, 108] proposed estimating a regression $y_i=\alpha+\beta x_i+u_i$ ($i=1, \cdots, n$) by ordering the observations so that $x_i \leqq x_{i+1}$, dividing them into two equal groups with the lower half in one and the upper half in the other, and then estimating β by $b=(\bar{y}_2-\bar{y}_1)/(\bar{x}_2-\bar{x}_1)$, where the bars denote arithmetic means in the two groups. Wald applied the technique to the case where both variables are measured with error. The authors assume that the x_i's are measured without error, and consider the efficiency of Wald's and similar methods of estimation relative to least squares. The results depend on the distribution of the x's. For the case when the x's have a Beta-distribution, it is shown that it is better to divide the observations so that the upper and lower groups contain each 30% of the observations; the middle 40% is disregarded. The efficiency is about .85. *K. J. Arrow* (Stanford, Calif.).

van Eeden, Constance. Maximum likelihood estimation of partially or completely ordered parameters. II. Nederl. Akad. Wetensch. Proc. Ser. A. **60**=Indag. Math. **19** (1957), 201–211.

This is a continuation of part I [same Proc. **60** (1957), 128–136; MR **18**, 773]. The first five sections are given in the earlier paper. Questions of consistency are considered in section 6 and examples are given in section 7. *Benjamin Epstein* (Stanford, Calif.).

van Eeden, Constance. Maximum likelihood estimation of partially or completely ordered parameters. Math. Centrum Amsterdam. Statist. Afdeling. Rep. **S 207 (VP 9)** (1956), 17 pp.

This report has been published and reviewed [Nederl. Akad. Wetensch. Proc. Ser. A. **60** (1957), 128–136; MR **18**, 773]. *Benjamin Epstein* (Stanford, Calif.).

Clark, Frank Eugene. Truncation to meet requirements on means. J. Amer. Statist. Assoc. **52** (1957), 527–536.

Foster, F. G. Upper percentage points of the generalized beta distribution. II. Biometrika **44** (1957), 441–453.

The author is concerned with tabulating the critical values for the distribution of the maximum root of a certain determinantal equation which arises in multivariate analysis. In part I [Foster and Rees, Biometrika **44** (1957), 237–247; MR **19**, 188] the tabulation was made for the case of two roots, and now is extended to the case of three roots. Values of x to 4D for which $P(O_{\max}<x)=P$ are given, for $P=.80(.05).95, .99$, $\nu_1=4(2)194$, $\nu_2=3(1)10$, where ν_1 and ν_2 are the d.f. occurring in the determinantal equation. *I. Olkin* (East Lansing, Mich.).

Muniruzzaman, A. N. M. On measures of location and dispersion and tests of hypotheses in a Pareto population. Calcutta Statist. Assoc. Bull. **7** (1957), 115–123.

The probability density function of the Pareto distribution is given as $f(x)=(\nu-1)w^{\nu-1}x^{-\nu}$ for $w \leqq x<\infty$, where the parameters are restricted to $w>0$, $\nu>1$. Letting X have a Pareto distribution, for $\nu \leqq 3$ the variance of X fails to exist. The author proposes in this instance the use of $\lambda=\nu/w$ as a measure of dispersion. Maximum likelihood estimates of λ, ν and their sampling distributions are discussed. *F. C. Andrews* (Eugene, Ore.).

Moore, P. G. The two-sample t-test based on range. Biometrika **44** (1957), 482–489.

This paper extends to the case of unequal sample sizes the two-sample t-test suggested by Lord [Biometrika **34** (1947), 41–67; MR **8**, 394] for testing the hypothesis of equality of means of two normal populations with common standard deviation σ, where the estimate of σ is based on the sample ranges. It is shown that not much power is lost by using the mean range as the estimate of σ instead of the more accurate weighted sum of the ranges as found in David [ibid. **37** (1951), 393–409; MR **13**, 668]. Included are (1) tables of the 10% 5% 2% 1% (twotailed) cutoff points for all combinations of sample sizes between 2 and 20, and (2) tables useful in computing the estimate of David. *T. S. Ferguson* (Los Angeles, Calif.).

Ray, W. D. Sequential confidence intervals for the mean of a normal population with unknown variance. J. Roy. Statist. Soc. Ser. B. **19** (1957), 133–143.

The asymptotic results of Anscombe [same J. **15** (1953), 1–21; MR **15**, 142] are supplemented by small-sample

analysis, with a somewhat modified sequential procedure. Illustrations based on artificial data. Comments on the problem of the "best" sequential procedure. *H. Wold.*

Šidák, Zbyněk; and Vondráček, Jiří. A simple nonparametric test of the difference of location of two populations. Apl. Mat. 2 (1957), 215–221. (Czech. Russian and English summaries)

Two ordered independent random samples $x_1<x_2<\cdots<x_n$; $y_1<y_2<\cdots<y_m$ are given. For solving the two sample problem the characteristic $Q=R+S$ is used, where R denotes the number of points from the sample $\{x_i\}$ which are less than y_1, and S is the number of points from the sample $\{y_j\}$ greater than x_n. Rosenbaum's test uses the number S only. The distribution of the characteristic Q is determined and an expression for the tail probabilities given. Tables of 1% and 5% critical values are given.

J. Janko (Praque).

Walsh, John E. Nonparametric mean estimation of percentage points and density function values. Ann. Inst. Statist. Math., Tokyo 8 (1957), 167–180.

For a random variable X with probability density $f(X)$, assumed to have a sufficient number of derivatives, let θ_p be defined by $\text{Prob}(X\leqq\theta_p)=p$ for given $0<p<1$, and let $X_1<X_2<\cdots<X_n$ be an ordered sample of X. Using methods introduced by F. N. David and N. L. Johnson [Biometrika 41 (1954), 228–240; MR 16, 382], the author shows how to determine an integer $2\leqq m\leqq 7$, subscripts $r(1), \cdots, r(m)$, and constants $a_1, \cdots, a_m, b_1, \cdots, b_m$, so that the linear combinations of order statistics $U=\sum_{i=1}^{m} a_i X_{r(i)}$ and $V=\sum_{i=1}^{m} b_i X_{r(i)}$ have the properties

$$E(U)=\theta_p+O(n^{-m/2}), \qquad E(V)=\frac{1}{f(\theta_p)}+O(n^{-(m-1)/2}).$$

Z. W. Birnbaum (Seattle, Wash.).

Fieller, E. C.; Hartley, H. O.; and Pearson, E. S. Tests for rank correlation coefficients. I. Biometrika 44 (1957), 470–481.

E. C. Fieller, T. Lewis and E. S. Pearson [Correlated random normal deviates, Cambridge, 1955; MR 17, 638] have recently published 3000 sets of correlated random normal deviates and a further 22,000 are available in manuscript. The present authors utilize the Fieller-Lewis-Pearson data in an empirical study of the measures r_S and r_K of rank correlation defined as follows. If a set of n objects is labelled 1, 2, 3, $\cdots$, n in accordance with the preferences of Judge A, let Judge B prefer them in the order $v_1, v_2, \cdots, v_n$ (so that the set of v's is a permutation of the first n integers). Then $r_S=1-6S_S/(n^3-n)$, where $S_S=\sum (i-v_i)^2$, and $r_K=4P_K/(n^2-n)-1$, where $P_K=\sum m_i$ and m_i is the number of j's with $j>i$ and $v_j>v_i$. Both r_S and r_K lie between $+1$ and -1, and ties do not arise in the present problem. Now consider a random sample (x_i, y_i) $(i=1, 2, \cdots, n)$ of paired observations from a bivariate normal population with correlation coefficient ρ. Let the pairs be arranged in order of increasing x's, and let $v_1, v_2, \cdots, v_n$ be the x-ranks of the ordered y's. With this postulated origin for the data, r_S and r_K become statistics for the population parameter ρ. Some analytical information is available on the distributions of these statistics; the authors study their behaviour in relation to artificial samples with $n=10$, 30, 50 and with $\rho=0.1(0.1)0.9$. It is found, for example, that $z_S=\tanh^{-1} r_S$ and $z_K=\tanh^{-1} r_K$ are approximately normally distributed with variances nearly independent of ρ; in fact

$$\text{var}(z_S)\simeq\frac{1.060}{n-3}, \quad \text{var}(z_K)\simeq\frac{0.437}{n-4}.$$

These numerical studies are being continued and further results will be reported in due course. *D. G. Kendall.*

Cooper, B. E. The effect of ties on the moments of rank criteria. Biometrika 44 (1957), 526–527.

The author gives a formula by means of which corrections can be obtained for moments of rank criteria when t individuals tie. *D. M. Sandelius* (Göteborg).

Pearce, S. C. Experimenting with organisms as blocks. Biometrika 44 (1957), 141–149.

The statistical implications of using organisms as blocks are considered, special attention being given to the case of a treatment applied to one plot having a local effect on the plot to which it is applied and remote effects elsewhere in the block, and also to the case of remote and local effects interacting. A worked example is given, and the usefulness of the method is discussed. (Author's summary.) *M. Zelen* (Washington, D.C.).

Zelen, Marvin. The analysis of covariance for incomplete block designs. Biometrics 13 (1957), 309–332.

The author reviews the derivation of the formulae for the intra-block analysis of covariance applicable to various experimental designs as a straightforward consequence of the general linear hypothesis. The author then deals with the recovery of inter-block information when the number of treatments is larger than the number of experimental units in a block. Formulae are given for application to balanced and partially balanced (2 associate classes) incomplete block designs. The author also outlines the appropriate covariance analysis for one concomitant variate when the regression coefficient is dependent on the block. A numerical example is included, illustrating the covariance analysis computations for a balanced incomplete block design with one concomitant variate. The data resulted from an investigation of the effects of four different geometrical shapes on the current noise of certain film-type composition resistors.

S. Kullback (Washington, D.C.).

Cusimano, Giovanni. I piani di sperimentazione con blocchi incompleti ed il metodo dei confronti fra osservazioni accoppiate. Statistica, Bologna 17 (1957), 443–481.

This is a comprehensive exposition of methods used in the analysis of paired comparisons. The basic model introduced by Bradley and Terry [Biometrika 39 (1952), 324–345; MR 17, 56] is as follows. Corresponding to t treatments there are t parameters $p_1, \cdots, p_t$ whose sum is unity, and the probability that the ith treatment will be preferred to the jth treatment in a paired comparison is given by $p_i/(p_i+p_j)$. The maximum likelihood equations are easily obtained and can be solved by iteration methods. The null hypothesis that all p_i are equal can then be tested against various alternatives by means of the likelihood ratio test.

The author is mainly concerned with taste preference tests, and discusses various modifications in the above described basic model, made necessary by this specific situation. The analysis of these modified models is discussed and tests, for various pertinent alternatives are given. Finally, the methods are also applied to data from educational statistics. *H. B. Mann* (Columbus, Ohio).

Méric, Jean. Sur le calcul de la fonction "OC" du test binomial de Wald, à partir de la relation de récurrence de Pólya. C. R. Acad. Sci. Paris **245** (1957), 1500–1502.

In the notation of the author's previous papers [same C. R. **241** (1955), 1255–1257, 1377–1380; MR **17**, 500], the *OC* function in that setting can be represented by a ratiónal function determined by the function K. [See Pólya, Univ. California Publ. Math. (N.S.) **1** (1948), 229–239; MR **10**, 312.] *J. Kiefer* (Ithaca, N.Y.).

Derman, C.; Littauer, S.; and Solomon, H. Tightened multi-level continuous sampling plans. Ann. Math. Statist. **28** (1957), 395–404.

The following generalizations of an inspection plan by Sieberman and Solomon [same Ann. **26** (1955), 686–704; MR **17**, 757] for continuous production are considered. The jth sampling level is said to be in use of every $(j-i)$th item is sampled. If i successive inspected items are free of defects, one adopts a higher level of sampling; if one defective item is found, one adopts a lower level of sampling. The previous paper moved sampling from the jth to the $(j+1)$st or $(j-1)$st level, while here one can move it to the $(j+s)$th or, on the other hand, to either the $(j-r)$th or the zeroth level (=100% inspection). Most of the paper is devoted to calculating the AOQ (average outgoing quality=fraction defective after sampling) in terms of the fraction defective in the original production process, and to calculating the AOQL (average outgoing quality limit=maximum possible fraction defective after sampling) in terms of the parameters of the plan, always assuming that the process is in statistical control. The results follow from certain properties of Markov chains.

S. W. Nash (Vancouver, B.C.).

Siraždinov, S. H. Concerning estimations with minimum bias for a binomial distribution. Teor. Veroyatnost. i Primenen. **1** (1956), 168–174. (Russian. English summary)

Let $f(p)$ be a polynomial of degree $n+1$, with highest coefficient unity. We can choose $g(m)$ $(m=0, 1, \cdots, n)$ so that

$$f(p)-\sum_{m=0}^{n} g(m)C_n^m p^m(1-p)^{n-m}=L(p, n+1),$$

where $L(p, n+1)=2^{-2n-1}T_{n+1}(2p-1)$ and $T_n(x)=\sum_{k=0}^{[n/2]}(-1)^k n(n-k)^{-1}C_{n-k}^k x^{n-2k}2^{n-2k-1}=n$th Chebyshev polynomial. Then if x is binomially distributed with probability p, $\max_{0\le p\le 1}|f(p)-Eg(x)|=2^{-2n-1}$, and $g(x)$ is the best uniform estimate of $f(p)$.

This result applies to acceptance sampling. Let a sample of n pieces be drawn from a lot of size N, with acceptance if c or less are defective, and 100 percent inspection otherwise. If p is the original lot fraction defective, and p^* the fraction defective in the accepted population, then

$$p^*=\begin{cases} p-x/N & x\le c,\\ 0 & x>c.\end{cases}$$

Put $Q(p)=Ep^*=\sum_{m\le c}(p-m/N)C_n^m p^m(1-p)^{n-m}$, $D(n,c)=2^{-2n+1}C_{n-1}^c$, and $h(x)=x/N$ for $x\le c$ and 0 for $x>c$. Define $q(x)$ by

$$q(0)=0,\ q(1)=n^{-1}(1-2^{1-2n}(-1)^c C_{n-1}^c),$$

$$q(m+1)=(-1)^{c+1}\frac{C_{n-1}^c}{C_n^{m+1}}B(n, m)+(-1)^m\sum_{k=1}^{m}(-1)^k C_{m+1}^k q(k)$$

for $1\le m\le c$, and

$$q(m+1)=(-1)^c\frac{C_n^m C_{m-1}^c-C_{n-1}^c}{C_n^{m+1}}B(n, m)+(-1)^m\sum_{k=1}^{m}(-1)^k C_{m+1}^k q(k)$$

for $c<m<n$, where

$$B(n, m)=(-1)^{n+m}2^{-2(n-m)}(n/m)C_{n+m-1}^{2m-1}.$$

Put $Q^+(x)=q(x)-h(x)+(x/n)D(n, c)$ and $Q^-(x)=q(x)-h(x)-(x/n)D(n, c)$. Then $EQ^-(x)\le Q(p)\le EQ^+(x)$ for $p\in[0, 1]$. Tables are given for $q(x)$ for $c=0$, 1 and n from 5 to 15. *A. A. Brown* (Wellesley, Mass.).

Keyfitz, Nathan. Estimates of sampling variance where two units are selected from each stratum. J. Amer. Statist. Assoc. **52** (1957), 503–510.

Nicholson, George E., Jr. Estimation of parameters from incomplete multivariate samples. J. Amer. Statist. Assoc. **52** (1957), 523–526.

Rider, Paul R. The midrange of a sample as an estimator of the population midrange. J. Amer. Statist. Assoc. **52** (1957), 537–542.

Stuart, Alan. The efficiency of the records test for trend in normal regression. J. Roy. Statist. Soc. Ser. B. **19** (1957), 149–153.

The 'records' test, proposed by Foster and Stuart for detecting a trend in a time-series, has its asymptotic efficiency compared with that of 4 other tests for trend in the case in which the alternative hypothesis is a normal regression. It is found that it has zero asymptotic efficiency compared with tests based on the rank correlation and rank serial correlation, but is infinitely more efficient than the difference sign and turning points tests.

C. A. B. Smith (London).

Aki, Keiiti. Space and time spectra of stationary stochastic waves, with special reference to microtremors. Bull. Earthquake Res. Inst. Tokyo **35** (1957), 415–456 (1 plate). (Japanese summary)

The author initially considers a wave system in one spatial dimension, and a decomposition of this system into components of given time-frequencies and velocities. He shows that the power of these components can conveniently be deduced by examining the spatial correlogram of a given frequency component, the frequency component being isolated by a filter. These results are generalized to the two-dimensional case, where they also permit the study of wave direction and polarisation. The author then gives an interesting account of the analysis, by these methods, of some microtremor records.

P. Whittle (Wellington).

See also: Probability: Gnedenko; Sukhatme; Breny. **Optics, Electromagnetic Theory, Circuits:** Blanc-Lapierre. **Economics, Management Science:** Arrow. **Programming, Resource Allocation, Games:** Bellman; Grenander.

PHYSICAL APPLICATIONS

Mechanics of Particles and Systems

Vâlcovici, Victor. Une extension des liaisons non holonomes et des principes variationnels. Ber. Verh. Sächs. Akad. Wiss. Leipzig. Math.-Nat. Kl. **102**, no. 4, 39 pp. (1958).

Le point de départ de l'auteur est la conviction que pour un système matériel une liaison doit être une obligation de nature géométrique imposée au déplacement du point et non à ses coördonnées ou à sa vitesse. Pour n points A_i, ayant le vecteur de position $\bar{r}_i$ il considère m relations de la forme

$$(1) \qquad \sum_{i=1}^{n} \bar{a}_{ij} d\bar{r}_i + \beta_j dt = 0 \quad (j=1, 2, \cdots, m),$$

où les vecteurs $\bar{a}_{ij}$ et les scalaires β_j sont des fonctions de t, $\bar{r}_1, \cdots, \bar{r}_n, \dot{\bar{r}}_1, \cdots, \dot{\bar{r}}_n, \bar{\gamma}_1, \cdots, \bar{\gamma}_n$, ainsi généralisant fortement la définition usuelle. Les relations (1) ont un second rôle dans la résolution du problème de mécanique, parce que dans l'équation fondamentale du travail virtuel

$$(2) \qquad \sum (\bar{F}_i - m_i \bar{\gamma}_i) \delta \bar{r}_i = 0,$$

il faut supposer $\sum \bar{a}_{ij} \delta \bar{r}_i = 0$, relations dérivées de (1). C'est pourquoi il faut se garder de remplacer (1) par une relation pour les vitesses $\dot{\bar{r}}_i$. Si $\beta_j = 0$ $(j=1, \cdots, m)$ le système est scléronome. Il y a des considérations analogues quand on introduit des coördonnées généralisées.

De (2) résultent les équations de mouvement connues $m_i \bar{\gamma}_i = \bar{F}_i + \sum \lambda_i \bar{a}_{ij}$, où λ_i sont des multiplicateurs; pour le travail élémentaire des réactions de liaison on a $d\Lambda - \sum \lambda_j \beta_j dt$; pour le cas scléronome on a $d\Lambda = 0$.

Après ces préliminaires l'auteur donne une discussion très remarquable des principes variationnels qui culmine d'un côté dans le théorème que, grace à la forme infinitésimale (1), la distinction entre les liaisons holonomes et non holonomes n'est pas essentielle et que les principes de moindre action et de Hamilton deviennent applicables aux deux categories. D'autre part il donne un principe intégral très général, dont résultent comme cas spéciaux non seulement les deux principes classiques, mais une infinité de principes analogues de la forme: $\int_{t_1}^{t_2} (\alpha_1 T + \alpha_2 U)\, dt$ est stationnaire, où α_1 et α_2 sont deux paramètres arbitraires; $\alpha_1 = 1$, $\alpha_2 = 0$ est le principe de moindre action, $\alpha_1 = \alpha_2 = 1$ est cel de Hamilton. *O. Bottema* (Delft).

Lur'e, A. I. On the theory of finite rotation of a rigid body. Prikl. Mat. Meh. **21** (1957), 571–573. (Russian)

Vector methods are used to show that two successive rotations Θ_1 (with axis the unit vector e_1 through an angle ϕ_1), Θ_2 are equivalent to Θ_2 followed by a rotation through an angle ϕ_1 about the axis into which e_1 is transformed by the rotation Θ_2. *R. C. T. Smith* (Armidale).

Martin, G. H.; and Spotts, M. F. An application of complex geometry to relative velocities and accelerations in mechanisms. Trans. A.S.M.E. **79** (1957), 687–691, discussion 692–693.

Sklianski, A. L. Sur les trajectoires singulières du problème général des trois corps avec le choc binaire. Ukrain. Mat. Ž. **9** (1957), 163–175. (Russian. French summary)

The title problem in its general form, assuming that the acting forces were analytic functions of mutual distances, $\lim_{r \to +0} r^{2k+1} f(r) = -2k < 0$ $(k \neq 0)$, was studied by Sokolov [Singular trajectories of a system of free material points, Akad. Nauk Ukrain. SSR, Kiev, 1951; MR **14**, 910]. In this paper, the author treats the same problem assuming $\lim_{r \to +0} r^{-2n+1} f(r) = l \neq 0$ $(n = -k)$. The system of differential equations of relative motion of mass-points is established. *D. Rašković* (Belgrade).

Vogel, Théodore. Sur des systèmes dynamiques à hérédité non linéaire et à mémoire totale. C. R. Acad. Sci. Paris **245** (1957), 1224–1226.

Collar, A. R. On the stability of accelerated motion: Some thoughts on linear differential equations with variable coefficients. Aero. Quart. **8** (1957), 309–330.

In studies of the stability of aeronautical systems, the resulting linear differential equations have coefficients which depend upon the flight speed, V. If V is constant, then the Routh-Hurwitz criteria are satisfactory for discussing the stability of motion. However, if V is a function of time, then no such simple procedure exists. Some approaches (mostly formal) to the discussion of stability when V is a function of time are suggested. The author stresses, in particular, inverse methods of approach in which particular forms of acceleration corresponding to prescribed solutions are derived. *J. K. Hale.*

★ **Павленко, В. Г. [Pavlenko, V. G.] Методы расчета бортовой качки судов. [Methods of calculation of the rolling of ships.]** Gosudarstv. Soyuz. Izdat. Sudostroit. Promyšl., Leningrad, 1956. 99 pp. 2.50 rubles.

A second approximation method of investigating nonlinear rolling oscillations of a ship is worked out. This method is used successively to solve the following problems of the theory of rolling ships: Ship oscillations in quiet water without taking drag into account; damping of ship oscillations in calm water; ship rolling in waves.

Consequently, methods have been obtained for computing the amplitude dependence of the free oscillation frequency of the ship, the dependence of the drag moment on the angular velocity and methods have been obtained for determining the basic parameters of ships rolling in waves (amplitude, phase, velocity and acceleration) which make the first approximation computation methods in present day use considerably more precise.

The methods proposed herein result in simple computational schemes and graphs which make possible their use in design organizations to compute ship rolling. Because of the extreme accuracy (in comparison with existing methods) and also because of the minor difficulties, these computational methods can be used to advantage to work out questions on standardization of the stability of maritime ships.

The method of investigating nonlinear oscillations presented herein can be used to analyze oscillations in other nonlinear systems with one degree of freedom in various branches of engineering, when the problem reduces to the solution of nonlinear differential equations similar to those considered herein.

The book is divided into: Introduction; Investigation of existing methods to compute ship rolling in waves; Investigation of free oscillations of a rolling ship without taking drag into account; Taking drag into account; Investigation of ship rolling in waves; Certain additional questions of the theory of ship rolling in waves. (From the author's foreword.) *M. D. Friedman.*

Fedorchenko, A. M. On the method of canonical averaging in the theory of nonlinear oscillations. Ukrain. Mat. Ž. 9 (1957), 220-224. (Russian. English summary)

Using the theory of canonical transformations, the author considers a method of canonical averaging for a mechanical system, described by a small parameter ε, the Hamiltonian of which periodically depends on the time. A generating function giving a transformation of variables where the transformed Hamiltonian does not explicitly contain the time is investigated. The solution of the successive approximations is found by expansion in a Fourier series.

The method is illustrated by an example of a spherical pendulum with a vibrating fulcrum. In the second approximation, all qualitative features of this motion are found. It is shown that under certain conditions for the frequency of the fulcrum vibrations the upper state of equilibrium becomes stable. The problem of small oscillations of the pendulum is also treated. *D. Rašković.*

Rumyancev, V. V. On stability of permanent rotations of a solid body around a fixed point. Prikl. Mat. Meh. **21** (1957), 339-346. (Russian)

In a previous paper [Prikl. Mat. Meh. **20** (1956), 51–66; MR **19**, 77] the author treated the stability problem of permanent rotations of a heavy solid. Using N. G. Chetaev's method for determining Liapunov's stability function, the approximate stability conditions of this motion in general and several special cases are deduced. Chetaev established [ibid. **20** (1956), 309–314; MR **18**, 128] the first quadratic integral of equations in variations of Laplace's three-body problem giving the possibility for solving the gyroscopic stability of this problem. In this paper, the author derives an analogous integral for equations in variations of the problem of permanent rotation of a solid body around a fixed point. The following cases are treated: (1) Lagrange's case ($A=B\neq C$, $x_0=y_0=0$, $z_0>0$); (2) S. V. Kowalevski's case ($A=B=2C$, $x_0>0$, $y_0=z_0=0$); and (3) the stability problem of rotation of a body around a principal axis through the center of gravity ($A\neq B\neq C\neq A$, $x_0\neq 0$, $y_0=z_0=0$).

D. Rašković (Belgrade).

Nemyckiĭ, V. V. On steady-state regimes in three-dimensional autonomous dynamical systems. Vestnik Moskov. Univ. Ser. Mat. Meh. Astr. Fiz. Him. **12** (1957), no. 1, 3–7. (Russian)

The author discusses the concept of steady-state regimes of dynamical systems and proposes to consider as such the almost-periodic motions, or rather the corresponding minimal sets K. He then shows that if the space is three-dimensional and K is asymptotically stable in the sense of Lyapunov then K is either an equilibrium point or the trajectory of a periodic motion or else the spectrum of the almost-periodic motions of K has just two generators.

J. L. Massera (Montevideo).

Garfinkel, Boris. On the motion of the earth satellite. Ordnance Computer Research Report, Ballistic Research Laboratories, Aberdeen Proving Ground, Md. vol. 4, (1957), no. 1, pp. 1–6. (Government Agencies, their contractors and others cooperating in Government research may obtain reports directly from the Ballistic Research Laboratories. All others may purchase photographic copies from the Office of Technical Services, Department of Commerce, Washington 25, D. C.) The gravitational potential of the earth is approximated by $V(r,\theta)=r^{-1}+cr^{-3}(3\sin^2\theta-1)$, where r is distance from the center, θ is latitude, c is a constant. The author proposes a particular splitting of V: $V=V_0+V_1$, where V_0 alone leads to an integrable problem, and then applies classical perturbation methods; he chooses V_0 of form $ar^{-1}+c_1r^{-2}(\sin^2\theta-c_2)$, where a, c_1, c_2 are constants to be chosen so as to eliminate secular perturbations. The solution of the unperturbed problem and first order perturbation effects are described without proof; a full exposition is promised in a paper to be published.

W. Kaplan (Ann Arbor, Mich.).

Hancock, M. The aiming problem in aerial gunnery. Aero. Quart. **8** (1957), 31–48.

The author discusses the calculations of the "lead angle" at which a moving aircraft gun must be pointed ahead of a moving target to obtain a hit. Known ballistics results concerning projectile motion are combined with a study of the kinematic aspects of the problem. The latter study, which the author claims to be new, takes into account the velocity of the aircraft and the angular velocity of the line of sight. *M. Marden.*

Lawden, Derek F. Maximum ranges of intercontinental missiles. Aero. Quart. **8** (1957), 269–278.

In an earlier paper [Aero. Quart. **6** (1955), 165–180], the author investigated the problem of programming the thrust direction of a rocket missile to achieve maximum range. In that paper the effects of the Earth's curvature and rotation, air resistance and variation of gravitational attraction with height above the Earth's surface were neglected. This paper continues the study by including effects of the Earth's rotation (in the equatorial plane) and the variation of gravitational attraction, while ignoring the rapidly negligible effects of air resistance. Numerical tables for optional inclination of rocket thrust, separately for trajectories with and against the Earth's rotation, are given. *A. A. Bennett* (Providence, R.I.).

v. Krzywoblocki, M. Z. On some static stability problems of a missile. I. Pakistan J. Sci. Res. **9** (1957), 126–132.

Sänger, Eugen. Zur Flugmechanik der Photonenraketen. Astronaut. Acta **3** (1957), 89–99.

See also: Ordinary Differential Equations: Ziemba. **Manifolds, Connections:** Haimovici. **Statistical Thermodynamics and Mechanics:** Casal. **Fluid Mechanics, Acoustics:** Moiseyev.

Statistical Thermodynamics and Mechanics

Casal, Pierre. Intégrale simple donnant le volume de l'espace des phases intérieur à la variété d'énergie constante. Application à la mécanique statistique. Loi de Gibbs. C.R. Acad. Sci. Paris **245** (1957), 1219–1221.

Bayet, Michel; Delcroix, Jean-Loup; et Denisse, Jean-François. Théorie cinétique des plasmas homogènes faiblement ionisés. IV. Étude de l'évolution de la partie isotrope de la fonction de distribution. J. Phys. Radium (8) **17** (1956), 1005–1009.

The results of part III [same J. (8) **17** (1956), 923–930; MR **18**, 611] are used to investigate the temporal behavior of the velocity distribution for an imperfect Lorenz gas, particularly the isotropic part. It is shown that in the

absence of an electric field, the distribution consists of a Maxwellian distribution plus some transients that decay exponentially with time. In the presence of an oscillatory electric field, the distribution consists of a stationary part, an oscillatory part and a transient part. For weak fields, the stationary part is still nearly Maxwellian with a slightly modified temperature, but for stronger fields this part may take on a quite different form. The paper also contains a discussion of the validity of the methods used and a comparison with other works. This, together with the earlier parts of this series, gives an extensive and penetrating analysis of weakly ionized gases.

G. Newell (Providence, R.I.).

Margenau, Henry. Conductivity of plasmas to microwaves. Phys. Rev. (2) **109** (1958), 6–9.

The author approximately solves the Boltzmann equation for the following situation: a microwave is passing through a neutral plasma containing electrons and a uniform positive background; the electron distribution in space and velocity changes little during one period of the wave; the collision cross section is a function of the velocity. The first approximation gives a relation between the derivation of the isotropic part and the current induced part of the distribution function. From this the complex conductivity is computed for three special choices of the isotropic part of the distribution function: all electrons have the same energy; the electrons are uniformly distributed in energy; the distribution is Maxwellian. It is shown that if the collision cross section is independent of the velocity, the result is in each case approximately identical with the Lorentz formula.

N. L. Balazs (Chicago, Ill.).

Hiroike, Kazuo. Radial distribution function of fluids.II. J. Phys. Soc. Japan **12** (1957), 864–873.

[For part I see same J. **12** (1957), 326–334; MR **18**, 836.] It is shown that the radial distribution function of fluid can be derived from the expression for the free energy in the form of a "functional derivative" with respect to the pair interaction potential. Three examples are treated on the line of the theory: (1) the simple chain approximation of classical fluid, (2) the second quantization and (3) the lowest state of the spinless bosons derived by Bogolyubov and Zubarev. A procedure, which differs from the usual one, is given in the derivation of the expressions by which the pressure and the internal energy are expressed in terms of the radial distribution function. (Author's summary.)

C. N. Yang (Princeton, N.J.).

Landau, L. D. Oscillations in a Fermi liquid. Soviet Physics. JETP **5** (1957), 101–108.

This is the second of a series of papers by Landau and his associates on the theory of excitations in a liquid of atoms satisfying Fermi statistics. The emphasis is on approximations which are expected to be valid at low temperatures, and the application to liquid He^3 is therefore in mind. In the first paper of the series [Soviet Physics. JETP **3** (1957), 920–925; MR **18**, 975] Landau derived a Boltzmann-type equation for the 'quasi-particles' (excitations) in a Fermi-liquid. In the paper now reviewed the propagation and absorption of various types of phonons is considered. The differential cross-section for the scattering of the quasi-particles is assumed to have a simple angular dependence and various interesting deductions are made. The first is that, although ordinary sound is strongly absorbed near absolute zero, another type of wave-disturbance called 'zero-sound' can be propagated. The velocity of zero-sound is calculated and found to be slightly greater than that of ordinary sound. The absorption is also calculated in the region somewhat above absolute zero. Finally, it is found that waves associated with the spin are likely to be strongly absorbed.

H. S. Green (Adelaide).

Meixner, J. Zur statistischen Thermodynamik irreversibler Prozesse. Z. Physik **149** (1957), 624–646.

The general formalism for irreversible processes is applied to an assembly of many independent 'systems'. Each system is described by a small number of variables (q, p), and the state of the total assembly is given by a distribution function $f(q, p)$ in phase space (μ-space). The number of systems $f(q, p)dqdp$ in any one phase cell $dqdp$ constitutes one variable of the type that is usually denoted by α_t. [See S. R. de Groot, Thermodynamics of irreversible processes, North Holland Publ. Co., Amsterdam, 1951; MR **13**, 307]. The conjugate 'forces' X_t are obtained by writing the rate of change of the entropy $s=-kf(f \log f)dqdp$ in terms of the 'fluxes' α_t, in this case $\delta f/\delta t$, assuming that f is not far from equilibrium. The usual 'Ansatz' for the phenomenological equations now amounts to assuming that $\delta f(q, p)/\delta t$ is linearly connected with the values of f at all other points (q', p') through an integral kernel $K(q, p; q', p')$. The Onsager relations then lead to a decomposition of K into a symmetric part describing irreversible phenomena and an antisymmetric part, describing reversible effects. The author makes the plausible assumption that the latter are identical to the ordinary Liouville flow in phase space [see also R. T. Cox, Rev. Mod. Phys. **24** (1952), 312–320; MR **14**, 522]. By specializing the symmetric part one obtains the equation of Kramers [Physica **7** (1940), 284–304; MR **2**, 140] for diffusion in phase space. As an example the case of electric dipoles in an external electric field is treated. {The author mentions the Boltzmann equation as another example, but it seems to the reviewer that this case has been excluded by assuming the particles independent.}

N. G. van Kampen (New York, N.Y.).

Frisch, Harry L. An approach to equilibrium. Phys. Rev. (2) **109** (1958), 22–29.

The approach to equilibrium of systems which undergo Poincaré cycles is discussed. Here the distribution functions are multiply periodic in time, but it is shown that phase space averages such as density, pressure, and temperature nevertheless approach limiting values over a sufficiently long period of time. Diffusion and heat conduction are discussed for the special case of a one-dimensional ideal gas where atoms are point masses constrained to move in a circle. It is found that this system obeys the expected transport equations, although the transport coefficients turn out to be proportional to the time.

S. Prager (Minneapolis, Minn.).

Saitô, Nobuhiko. Irreversible processes and fluctuations. J. Phys. Soc. Japan **12** (1957), 1321–1326.

This is mostly a restatement of the results of L. Onsager and S. Machlup [Phys. Rev. (2) **91** (1953), 1505–1512; MR **15**, 273] and N. Hashitsume [Progr. Theoret. Phys. **8** (1952), 461–478; MR **14**, 1048].

D. Falkoff.

Elasticity, Plasticity

★**Swainger, Keith. Analysis of deformation. III. Fluidity.** Chapman & Hall, Ltd., London, 1956. xxvii+266 pp. $13.00.

This is a very original book, bearing little resemblance to any other treatment of fluid motion. This volume, like the author's two preceding on solids [1954; MR **16**, 307, 764], the reviewer finds bewildering. The author presents a new theory of fluid behavior. In searching through the many pages of formulae it is difficult to find those defining this new theory. Judging by pp. 31–32, the reviewer hazards the conjecture that it is of the type

strain rate=linear function of stress rate and stress,

and thus if formulated correctly might be included among the hygrosteric materials of Noll [J. Rational Mech. Anal. **4** (1955), 3–81; MR **16**, 764]. A particular feature of the author's view, however, is a boundless enmity toward the non-linear. He devotes Chapter III to an exposition of the classical theory and a critique of Euler and all who have followed him in considering the acceleration to be $a=\partial v/\partial t+v\cdot\text{grad } v$. According to the author, the acceleration which should enter the equations of motion is $\partial v/\partial t$. The author regards as a significant support of his view the fact that according to the Navier-Stokes equations, a simple tensile stress cannot produce a steady motion of simple extension.

The author uses extensively the results from his earlier volumes, not correcting the major errors pointed out by Richter [Zbl. Math. **56** (1955), 175–176] and others. While he parades his historical erudition, it seems to be derived mainly from secondary sources, since he attributes to himself, recording the dates when he discovered them, numerous results that can be found in the literature he cites. *C. Truesdell* (Bloomington, Ind.).

Smith, R. C. T. Tension of an infinite plate cut along a circular arc. J. Math. Phys. **36** (1957), 223–233.

Brdička, Miroslav. On the general form of the Beltrami equation and Papkovich's solution of the axially symmetrical problem of the classical theory of elasticity. Czechoslovak J. Phys. **7** (1957), 262–274. (Russian summary)

The author shows how to get the explicit form of the Beltrami-Michell equations in orthogonal curvilinear physical components by specialization from their contravariant form. He then shows how to derive the Papkovitch solution for axially symmetrical problems by specialization of the general solution of the equilibrium equations. {Since the author makes a point of correcting attributions, the reviewer remarks that what the author calls the Trenin solution (1953) is due essentially to Love [Elasticity, 2nd ed., Cambridge, 1906, § 188] and what he calls the Finzi-Krutkov solution (1934, 1949) is due to Gwyther [Mem. Proc. Manchester Lit. Philos. Soc. **56** (1911–1912), no. 10.} *C. Truesdell.*

Rozenberg, L. A. Direct variational methods in a mixed problem of the theory of elasticity for a half-space. Akad. Nauk Uzbek. SSR. Trudy Inst. Mat. Meh. **13** (1954), 71–91. (Russian)

The boundary value problem in question consists of the determination of the displacement vector $u(x, y, z)$, twice continuously differentiable on the half space $z>0$, and satisfying the system

$$(\lambda+\mu)\text{ grad div } u+\mu\Delta u=\rho K$$

plus the boundary conditions (on $z=0$):

$$u|_\omega=\varphi(x, y),\ t_n|_{\text{outside }\omega}=\psi(x, y),$$

where λ and μ are Lamé's constants of elasticity, ω is a given smooth subset of $z=0$, φ and ψ are given functions, and t_n denotes the surface tractions. The problem is considered as a problem in Hilbert space, involving the determination of a vector u satisfying $Au=K$, with A a suitable linear operator. The treatment by direct methods follows then along the general lines laid down by L. Liusternik and V. I. Sobolev [Elements of functional analysis, Gostehizdat, Moscow-Leningrad, 1951; MR **14**, 54], S. G. Mihlin [Direct methods in mathematical physics, Gostehizdat, Moscow-Leningrad, 1950; MR **16**, 41], and S. L. Sobolev [Some applications of functional analysis in mathematical physics, Izdat. Leningrad. Gos. Univ., 1950; MR **14**, 565]. *J. B. Diaz.*

Solyanik-Krassa, K. V. Stress functions of an axisymmetric problem in the theory of elasticity. Prikl. Mat. Meh. **21** (1957), 285–286. (Russian)

Relationships are exhibited among those functions, satisfying the harmonic, biharmonic or some closely related equation in cylindrical coordinates, in terms of which solutions to problems of symmetrical strain in solids of revolution have been independently constructed by Michell, Love, Neuber et al. and the author.

R. N. Goss (San Diego, Calif.).

Vocke, Wolfgang. Eine geschlossene Lösung bei Torsion zylindrischer Stäbe mit Umlaufkerbe, Ermittlung von Drillsteifigkeiten. Z. Angew. Math. Mech. **37** (1957), 409–415. (English, French and Russian summaries)

Chattarji, P. P. Twisting of a hollow circular cylinder of cylindrically aelotropic material with outer surface fixed and inner surface acted on by tangential tractions. Indian J. Theoret. Phys. **4** (1956), 59–64.

Assuming that the ends of the cylinder are free of stress, the author uses a known type of series solution of the differential equation

$$\frac{\partial^2 v}{\partial r^2}+r^{-1}\frac{\partial v}{\partial r}-r^{-2}v+k^2\frac{\partial^2 v}{\partial z^2}=0$$

to state the solution of his problem in its dependence on the Fourier coefficients of the circumferential traction distribution. *E. Reissner* (Cambridge, Mass.).

Gladwell, G. M. L. Uniqueness theorems for thin clamped anisotropic plates. Mathematika **4** (1957), 70–75.

This paper deals with problems of transverse displacements of thin anisotropic plates. Proofs of uniqueness of solution under certain conditions are given for problems of plates occupying both finite and infinite regions.

A. E. Green (Newcastle-upon-Tyne).

★**Girkmann, Karl. Flächentragwerke: Einführung in die Elastostatik der Scheiben, Platten, Schalen und Faltwerke. 4te Aufl.** Springer-Verlag, Vienna, 1956. xix+596 pp. DM 66.00.

This fourth edition of a major reference work on the theory of plates and shells consists of an enlargement and expansion of the earlier editions. Orthotropic plate theory has been integrated into the body of the text (having been

dealt with earlier in an Appendix), improved forms of solutions particularly convenient for calculation purposes are emphasized, and a variety of new topics has been introduced. Of special interest is the detailed treatment of various forms of shell structures composed of curved and flat plate elements.

The coverage of this reference text is impressive. Plates loaded in their planes and bent plates are comprehensively dealt with, starting with the basic theory and proceeding to the solution of practical problems. Membrane theory and bending theory for shells are exhaustively treated in similar fashion. Buckling of plates and shells is also covered, though less thoroughly. Throughout the text, emphasis is placed on the utilization of the results in structural design practice.

For graduate level engineering instruction and for the design specialist, the text can be endorsed without qualification as an authoratitive and valuable treatise. The author's style combines mathematical facility with a lucid physical understanding of the subject, a virtue which is becoming increasingly rare in the modern concentration on mathematical adeptness alone.

M. Goland (San Antonio, Calif.).

★ **Вол′мир, А. С. [Vol′mir, A. S.] Гибкие пластинки и оболочки. [Flexible plates and shells.]** Gosudarstv. Izdat. Tehn.-Teor. Lit., Moscow, 1956. 419 pp. 15.20 rubles.

This is a very comprehensive account of the buckling and large deflection theory of flat and cylindrical panels, cylindrical and spherical shells. There are 220 references, about half being on the non-Russian literature, and all the important papers seem to have been included. The main impression the book gives is of extreme thoroughness. Most of the solutions of particular problems are obtained by approximate methods (Bubnov-Galerkin, Rayleigh-Ritz, etc.) and the results are always presented in almost immediately usable form, e.g. as a graph, table or simple formula. Experimental results are discussed in some sections and there are some interesting photographs of buckled shells. *R. C. T. Smith* (Armidale).

Bassali, W. A.; and Dawoud, R. H. Green's functions for thin isotropic plates containing holes. Proc. Cambridge Philos. Soc. **53** (1957), 755–763.

This paper is concerned with the small transverse displacement of an infinite thin plane isotropic plate due to the application of a transverse force applied at an arbitrary point of the plate. The plate has its outer edge free, and is clamped along and bounded internally by a closed curve that can be mapped onto the unit circle by means of a polynomial. Three polynomials are considered and in each of these cases the deflexion is obtained in finite terms. Circular and elliptic holes as well as curvilinear polygonal holes are included as special cases. (Authors' abstract.) *R. M. Morris* (Cardiff).

Bassali, W. A. The transverse flexure of thin perforated elastic plates supported at several points. Proc. Cambridge Philos. Soc. **53** (1957), 744–754.

In this paper exact solutions in finite terms are obtained for the problem of an infinitely large plate with outer edge free and an inner free circular boundary, the plate being supported at any number of interior or boundary points and normally loaded over a circle. The load considered is symmetrical with respect to the centre of the circle. (Author's abstract.) *R. M. Morris.*

Bassali, W. A. The transverse flexure of thin elastic plates supported at several points. Proc. Cambridge Philos. Soc. **53** (1957), 728–743.

This paper depends upon the method developed by Kolossoff and Muskhelishvili for problems of plane elasticity and later extended to plate problems by Lechnitzky. Exact solutions in closed forms are obtained for the problem of a thin circular plate supported at several interior or boundary points and normally loaded over the area of an eccentric circle, the load being symmetrical with respect to the centre of the circle and the boundary of the plate being free. Explicit formulae for the deflexion, the bending and twisting moments and shearing stresses are given at any point of the plate. As limiting cases plates in the form of the infinite plane and half plane are also considered. (Author's abstract.) *R. M. Morris.*

Remnev, Yu. I. Problem of the bending of a thin rectangular plate with three clamped sides and one free side. Vestnik Moskov. Univ. Ser. Mat. Meh. Astr. Fiz. Him. **12** (1957), no. 1, 35–38. (Russian)

Vyrbanov, H. P. Integration of systems of linear differential equations in the theory of prismatic and slightly pyramidal shells of V. Z. Vlasov. Izv. Akad. Nauk SSSR. Otd. Tehn. Nauk **1957**, no. 6, 64–76. (Russian)

Sekhri, J. N. Some problems relating to stresses in an infinite plate containing circular holes. Proc. Nat. Inst. Sci. India. Part A. **23** (1957), 246–252.

Fung, Y. C.; Sechler, E. E.; and Kaplan, A. On the vibration of thin cylindrical shells under internal pressure. J. Aero. Sci. **24** (1957), 650–660.

The natural frequencies of the "breathing modes" of thin-walled circular cylinders subjected to internal pressure are considered. A frequency equation is derived from the standard equations of thin-shell theory. A simplified expression for the frequency is obtained if only the radial component of the inertia forces is considered, as suggested by E. Reissner [Ramo-Wooldridge Corp., Aeromech. Rep. no. **AM 5–6** (1955)]; then the resulting equations are Marguerre's equations for slightly curved plates. It is shown that for very thin shells, with freely-supported ends and not subjected to internal pressure, the Reissner assumption has negligible effect on frequency, except for the lowest values of the numbers of circumferential waves, n. For such shells the lowest frequencies correspond to relatively high values of n [R. N. Arnold and G. B. Warburton, Proc. Roy. Soc. London. Ser. A. **197** (1949), 238–256 . Using the Reissner assumption these frequencies are shown to increase rapidly with increasing internal pressure p; also, as p increases, the value of n for which the frequency of the shell is a minimum decreases rapidly. *G. B. Warburton.*

Volterra, E.; and Zachmanoglou, E. C. Free and forced vibrations of straight elastic bars according to the „method of internal constraints". Ing.-Arch. **25** (1957), 424–436.

In previous papers [Ing.-Arch. **23** (1955), 402–409, 410–420; **24** (1956), 317–329, 392–400; MR **17**, 556, 557; **18**, 526, 690], the first author has used his method of "internal constraints" to express the solution of vibration problems for curved and twisted bars in terms of the eigenvalues and eigenfunctions of the corresponding straight

untwisted bars. In the present paper the lateral, longitudinal, and torsional vibrations of straight uniform bars are treated from the point of view of the method of "internal constraints". Effects of transverse shear and rotatory inertia are included, and several types of end conditions are handled. Expressions are obtained for the frequencies and modal shapes. *W. E. Boyce* (Troy, N.Y.).

Shinbrot, Marvin. On the analysis of linear and nonlinear systems. Trans. A.S.M.E. **79** (1957), 547–551, discussion 551–552.

★**Pignedoli, Antonio. Sulle vibrazioni di una piastra ellittica incastrata all'orlo.** Scritti matematici in onore di Filippo Sibirani, pp. 201–217. Cesare Zuffi, Bologna, 1957.

Newman, E. G.; and Mindlin, R. D. Vibrations of a monoclinic crystal plate. J. Acoust. Soc. Amer. **29** (1957), 1206–1218.

The eigenvalues for this problem can be obtained by solving Ekstein's highly complicated, transcendental characteristic equations. After rederiving these equations, the authors show how one goes about obtaining an approximate but detailed description of the eigenvalues by considering lattices of curves corresponding to simplified versions of this problem ("mixed boundary conditions") and such that the curve intersections correspond to roots of the exact equations. This method, which has been developed over the last six or seven years by R. D. Mindlin and collaborators [see, e.g., MR **19**, 699], is a truly significant step forward in one's physical and mathematical understanding of the difficult vibration problems that occur in practice, e.g., the present one. *I. Tolstoy.*

Pekeris, Chaim L.; and Lifson, Hanna. Motion of the surface of a uniform elastic half-space produced by a buried pulse. J. Acoust. Soc. Amer. **29** (1957), 1233–1238.

In this paper an exact solution is obtained of the equations governing the motion of a uniform isotropic semi-infinite elastic solid when a concentrated vertical force, whose time-variation is represented by a Heaviside unit function, is applied at a depth H below the surface of the solid. Such a source emits both P-waves and S-waves. The expressions for the components of the surface displacement are obtained in the form of single integrals over a fixed range, and numerical values obtained by means of the electronic computer of the Weizmann Institute (WEIZAC). A discussion of the results of these numerical investigations is given, the implications for seismology being emphasized. *I. N. Sneddon* (Glasgow).

Datta, A. N. Longitudinal propagation of elastic disturbance for linear variations of elastic parameters. Indian J. Theoret. Phys. **4** (1956), 43–50.

The propagation of plane waves in an elastic medium is considered. The treatment throughout is one-dimensional, it being assumed that the space coordinate is x and the time coordinate is t. It is assumed that the medium is isotropic, but that Lamé's elastic constants λ and μ vary according to the laws $\lambda=\lambda_0+\lambda_1 x$, $\mu=\mu_0+\mu_1 x$. A general solution of the equation of motion is obtained by first transforming from x to a new variable $2c\omega[\lambda_0+2\mu_0+(\lambda_1+2\mu_1 x)]^{\frac{1}{2}}$. By means of this general solution and the theory of Laplace transforms, solutions of the following three problems are obtained: (1) The end $x=0$ is free from stress and at the end $x=H$ there is a source producing periodic displacement; (2) the end $x=0$ is free from stress and at the end $x=H$ there is a source producing periodic stress; (3) the end $x=0$ is free from stress and an impulsive stress is applied at the end $x=H$. A discussion of the simplifications resulting from small variations in the elastic constants is also given, in particular the case in which the Poisson ratio is constant throughout the solid but the Young's modulus varies linearly with x. *I. N. Sneddon* (Glasgow).

Huan Tun. Reflection and refraction of a plane longitudinal elastic wave from the bottom of the ocean. Vestnik Moskov. Univ. Ser. Mat. Meh. Astr. Fiz. Him. **12** (1957), no. 1, 22–34. (Russian)

The author considers the problem of the reflection and refraction of longitudinal plane waves from the bottom of the sea. The problem is idealized to that of a semi-infinite elastic solid covered by a layer of heavy liquid. It is supposed that a plane longitudinal elastic wave is moving with constant velocity a^{-1} through the solid in such a way that the wave front impinges at an angle θ on the plane boundary. Instead of fixed coordinates x_1 and y_1, the author uses a system of moving (and dimensionless) coordinates $x=(x_1+mt/a)H^{-1}$, $y=y_1/H$, where H is the depth of the liquid layer and $m=\operatorname{cosec}\theta$. A general solution of the equations of motion of the liquid layer and the elastic solid is derived in terms of the parameters $\zeta=a^2gH$, $\xi=gH/c^2$, where g is the acceleration due to gravity and c is the velocity of sound in the liquid layer. The properties of this solution are examined analytically especially for the case in which $1-m^2/a^2c^2>\xi$. *I. N. Sneddon* (Glasgow).

Močalov, S. D. Propagation of elastic-plastic waves along a rod of variable limit of elasticity. Tomskiĭ Gos. Univ. Uč. Zap. Mat. Meh. **25** (1955), 49–67. (Russian)

The author considers a semi-infinite rod composed of a work-hardening material which obeys the laws of Prandtl for repeated loading and unloading. The elastic limit at any point in such a rod is a function of the residual strain at that point, and the residual strain is in turn a function of the prior history of plastic straining of the rod. For any monotonic loading process the elastic limit is determinable as a function of space, the space coordinate being measured as the distance from the end of the rod.

The most commonly encountered situation is one in which prior loading cycles have left the absolute value of the elastic limit a decreasing function of the distance from the end of the rod. The propagation of elastic-plastic waves along such a rod of decreasing elastic limit is discussed in considerable detail, both for monotonically increasing longitudinal loads and for the limiting case of longitudinal impact loading. The theory of characteristics is used to analyse the general properties of the strain and particle velocity distributions and to indicate a method for complete numerical solution of the problem. One interesting fact which emerges is that under an impact strong enough to cause plastic deformation of the end of the rod, there will be a region of plastic deformation, directly behind the initial elastic wave, in which the strains and particle velocities do not depend either on the intensity of the impact or on the subsequent loading pattern.

The case of a rod with an increasing elastic limit is also discussed but only for a monotonically increasing load. This problem can also be treated successfully by the

method of characteristics, but the numerical work is more arduous, the chief complicating factor being the difficulty of locating the elastic-plastic boundary in the space-time plane.

In both instances only the process of loading is considered. Some of the difficulties involved in extending the analysis to waves of unloading are discussed briefly.

P. Mann-Nachbar (Palo Alto, Calif.).

Močalov, S. D. Repeated impact on the end of a half-infinite rod. Tomskiĭ Gos. Univ. Uč. Zap. Mat. Meh. **25** (1955), 68–76. (Russian)

It is assumed that the rod is composed of a material which exhibits work-hardening. In the plastic range, the slope of the stress-strain curve, $d\sigma/d\varepsilon$, is a non-increasing function of the strain, ε, and is always smaller than the elastic modulus. It is further assumed that the laws of Prandtl are followed for repeated loading and unloading and that there is no dissipation of energy during a cycle. The elastic limit depends on the residual strain and increases after every impact which produces plastic deformation.

It is noted that the problem of repeated impact on such a rod can be reduced to the problem of a single impact on a rod for which the elastic limit is a decreasing function of the distance from the struck end. The latter problem is discussed by the author in some detail in the paper reviewed above.

Several additional results are deduced concerning the elastic limits and residual strains at the struck end of the rod when all impacts starting with the nth, say, are delivered with the same velocity. It is shown that if, for such a loading pattern, the nth impact results in plastic deformation and, therefore, in an increase of the elastic limit at the end of the rod, then all subsequent impacts will also result in plastic deformation and additional increase of the elastic limit. If, now, for the material under consideration, $\lim d\sigma/d\varepsilon \neq 0$ as $\varepsilon \to 0$, then both the sequence of elastic limits and the sequence of total residual strains thus formed are bounded from above and approach finite limits as the number of impacts increases indefinitely. The explicit forms of these limits are derived by the author.

A material exhibiting linear work-hardening is used as an example to illustrate the above results. It is indicated that if suitable restrictions are placed on the impact velocity, these results remain valid when the stress-strain relationship is such that the stress is bounded from above for increasing strain. *P. Mann-Nachbar.*

Klyušnikov, V. D. On the laws of plasticity for a particular class of loading paths. Prikl. Mat. Meh. **21** (1957), 533–543. (Russian)

"There is a certain group of phenomena which are observed in experiments on the plastic strain of metals which are not accounted for by the existing phenomenological theories of plasticity. To such phenomena belong, for example, the effect of sharp break of the strain path in experiments on extension and torsion of thin-walled tubes and secondary phenomena accompanying this effect, also phenomena of change of plastic strain in that process of loading in which the second invariant of the stress deviator is unchanged; in addition the value of the instantaneous bending modulus G_i, which apparently plays a large part in problems of stability, in certain experiments differs markedly both from the value given by the isotropic theory of plastic flow for which G_i is always equal to the elastic bending modulus G_0 and from the value

$$G_i = \frac{G_0}{1+3G_0(1/E_s - 1/E_0)} \tag{1}$$

given by the isotropic theory of strain. (Here E_0 and E_s denote the elastic and secant moduli of the tensile stress-strain curve.) In actual fact, G_i lies between these two values.

In the present article, on the basis of certain assumptions for a particular class of loading paths, a law is derived connecting stresses and strains which ultimately accounts in general terms for the phenomena mentioned above, and gives a value for G_i lying between G_0 and G_i as in Eq. (1). From this law, as a particular case, follows the simple isotropic theory of plastic flow, which is sometimes called Laning's law." (From author's summary.)

The difficulties to which the author refers are well-known, particularly in relation to the serious shortcomings of plastic flow theories of stability. In this particular case, discrepancies between experimental results and theoretical predictions for critical loads are often attributed to initial imperfections, and in fact some of the available theoretical work strongly supports this viewpoint. The present paper apparently seeks a more fundamental basis for the explanation of such discrepancies, together with those of a related nature. *H. G. Hopkins.*

Feĭnberg, S. M. Plastic flow of a curved shell for the axisymmetric problem. Prikl. Mat. Meh. **21** (1957), 544–549. (Russian)

A long curved plate (part of a circular cylindrical shell) is simply supported at its long, straight, parallel edges. The convex side of the plate is subjected to a uniformly-distributed load. Assuming that the material is ideal-plastic rigid, the distribution of stress in the plate at collapse is discussed via the usual simplifying assumptions of conventional engineering shell theory. Particular attention is given to the case when Tresca's yield condition is satisfied, and the collapse load is determined.

The paper gives no discussion of the mode of failure of the curved plate. *H. G. Hopkins* (Fort Halstead).

Drucker, D. C.; and Shield, R. T. Bounds on minimum weight design. Quart. Appl. Math. **15** (1957), 269–281.

As the basic problem of design for minimum weight, the authors consider a volume within which a structure loaded by a given force distribution is to be placed. The loads are to be supported by some distribution of a given elastic perfectly-plastic material in this volume. The problem is to determine a structure or distribution of the material which is just at the point of collapse under the loads and which involves minimum volume of material. If the material is homogeneous, then the problems of design for minimum volume and weight coincide. The approach to the general three-dimensional problem is made via the two theorems of limit analysis, and upper and lower bounds on the minimum volume are obtained. The analysis allows for the presence of body forces and inertial forces due to steady rotation. Formal modifications of the present analysis also allow for inhomogeneous material. A similar, but independent, analysis is then given for a plane sheet of variable thickness under plane stress conditions, and the nature of the extension to general membranes is briefly indicated. Next, it is proved that in most cases it is not possible to design a body compatible with a collapse state in which the modified dis-

sipation rate $\Delta=\sigma_{ij}\varepsilon_{ij}^{p}-F_i u_i$ (in the usual notation) is constant. However, when this is possible, a minimal design is obtained. In other cases, the general approach must be used to obtain either the minimum design or bounds on its volume. A thin homogeneous plate, treated in three dimensions, is an example for which Δ is not constant in the minimal design. A sandwich plate with lightweight core is a contrary example, Δ then being constant in the two identical face sheets. Attention is now given to the solution of illustrative examples of the present problem. First, the case of a circular annular disc, rotating uniformly in its own plane about its centre, and loaded radially and uniformly at its outer edge is discussed. The material is assumed to obey Tresca's yield criterion, and minimum volume designs are obtained for discs of unlimited or restricted thickness. In the former situation only is Δ a constant. Second, the case of a simply-supported circular sandwich plate of constant core thickness subjected to uniformly-distributed transverse load is discussed. The problem here is to design the variable thickness of the two identical face sheets to have minimum volume. This is considered for a general yield criterion, and specialization is then made to Tresca's yield criterion, for which an explicit and relatively simple solution obtains. The quantity Δ is of course constant.

The present paper extends considerably previous work by the same authors [to be published in Proc. IX Inter. Cong. Appl. Mech., Brussels, 1956]. It should be remarked that although minimum weight designs of engineering structures may not always be completely practical, or even lead to particularly worthwhile economies in comparison with more conventional designs, their importance lies perhaps primarily in setting up standards of comparison. *H. G. Hopkins* (Fort Halstead).

Kačanov, L. M. Elastic-plastic equilibrium of a wedge under conditions of plane stress. Prikl. Mat. Meh. **21** (1957), 413–418. (Russian)

A thin uniform wedge of indefinite extent is subjected to uniformly-distributed load applied normally to one edge and in the plane of the wedge. The wedge is in equilibrium (instability is supposed not to occur), and the applied edge forces are supposed suitably reacted at infinity. Subject to the assumptions of plane stress and perfectly-plastic elastic material obeying von Mises' yield criterion, an analysis of this problem is given for arbitrary wedge angle. The same problem has been treated by P. M. Naghdi [Technical Report, Engineering Research Institute, University of Michigan, Ann Arbor, May 1957] assuming Tresca's yield criterion and with the wedge angle not exceeding 90°. In the latter case the problem is statically-determinate, but in the former case this is not generally true. The relative simplicity of the present problem (just as with the corresponding problem of plane strain) is directly due to the fact that no fundamental linear dimension is involved. All stresses and strains (but not displacements) are therefore functions only of an angular co-ordinate measured about the wedge apex. *H. G. Hopkins* (Fort Halstead).

Olszak, W.; and Murzewski, J. Elastic-plastic bending of non-homogeneous orthotropic circular plates. I. Arch. Mech. Stos. **9** (1957), 467–485. (Polish and Russian summaries)

A thin circular plate is fixed at its circumference and subjected to transverse loading varying only with the radial distance r from the centre. The deflections are restricted to be small compared with the thickness. The physical properties are allowed to depend on r. The material is non-hardening, with a quadratic or linear yield condition. In the plastic zone the elastic component of strain is disregarded. This appears to be responsible for the discontinuity in stress at the elastic-plastic interface in the first attempted solution. In a second attempt the stress is made continuous at the expense of abandoning the identity of the plastic potential and yield function.

A satisfactory solution when the material is homogeneous was previously obtained by F. A. Gaydon and H. Nuttall [J. Mech. Phys. Solids **5** (1956), 62–65; MR **18**, 351]. *R. Hill* (Nottingham).

Ivlev, D. D. Approximate solution of problems of small elastic-plastic deformations. Dokl. Akad. Nauk SSSR (N.S.) **113** (1957), 527–528. (Russian)

Approximations of nth order for stresses (two-dimensional problem, polar coordinates) are used with equilibrium conditions to arrive at a solution with terms of the form $\rho^k \sin m\theta$. A fourth degree polynomial characteristic equation determines k, with m and χ as parameters, where $\sigma_i=Ae_i^{\chi}$. The author notes the special case for $m=1$ treated by L. M. Kačanov [Izv. Akad. Nauk SSSR. Otd. Tehn. Nauk **1956**, no. 9, 65–71; MR **18**, 526]. *R. E. Gaskell.*

Hu, L. W. Studies on plastic flow of anisotropic metals. J. Appl. Mech. **23** (1956), 444–450.

Hill's theory of plastic flow of anisotropic metals is extended to include strain-hardening and applied to plane-stress and plane-strain problems, to thickwalled cylinders under internal pressure and to the interpretation of biaxial tension-tension and tension-torsion tests.

Šesterikov, S. A. On a variational principle in the theory of creep. Izv. Akad. Nauk SSSR. Otd. Tehn. Nauk **1957**, no. 2, 122–123. (Russian)

Biot, M. A. New methods in heat flow analysis with application to flight structures. J. Aero. Sci. **24** (1957), 857–873.

In this important paper the author presents some variational principles in heat flow analysis with numerous applications. The concepts of thermal force, dissipation function and thermal potential are introduced, leading to ordinary differential equations of the Lagrangian type for the thermal flow-field. The variational principles include boundary-layer conduction, non-linear phenomena with temperature dependent parameters, and radiation. The methods lead to drastic simplifications in calculating transient heat flow in complex structures and are in part a direct application of some general variational principles developed earlier for linear thermodynamics [J. Appl. Phys. **25** (1954), 1385–1391; **27** (1956), 240–253; Phys. Rev. (2) **97** (1955), 1463–1469; MR **17**, 1035; **16**, 1189]. *E. H. Mansfield* (Farnborough).

Nowacki, Witold. Two steady-state thermoelastic problems. Arch. Mech. Stos. **9** (1957), 579–592. (Polish and Russian summaries)

Consider an elastic half-space with the boundary B. Let B_1 be a circular subregion of B and let B_2 be the complement of B_1 with respect to B. Assume that B is free from surface tractions and that B_2 is perfectly insulated. This paper aims at the steady-state thermal stresses which arise if B_1 is exposed to a uniform heat-input (Problem 1) or if B_1 is maintained at constant temperature (Problem 2).

The solutions obtained are in terms of improper integrals involving Bessel functions. For Problem 1, an alternative representation by means of infinite series of hypergeometric functions is deduced. In both problems, the ensuing thermal-stress field is found to be plane and parallel to the boundary. This conclusion, it is claimed, continues to hold even if B_1 is an arbitrary subregion of B and the boundary values on B_1 are not necessarily constant. The generalized conclusion is established only for Problem 1, with the aid of the solution appropriate to a point-source of heat on B. {It is not clear why the author re-derives this known elementary solution [See E. Melan and H. Parkus, Wärmespannungen, Springer, Berlin, 1953, which is referred to in the paper.]} No numerical results are given.

The special problems treated here are closely related to the general steady-state problem for the half-space which is encountered if the temperature is prescribed at will over an arbitrary subregion B_1 of B, and is constant (zero) over the complement B_2. The solution to the latter problem [Quart. Appl. Math., **14** (1957), 4], which exhibits the same degeneracy of the stress distribution, is curiously ignored by the author. Yet, the potential theoretic scheme employed in the paper cited is immediately applicable to the particular problems considered at present and yields closed solutions in terms of elliptic integrals, the numerical evaluation of which presents no difficulties.

E. Sternberg (Providence, R.I.).

Ignaczak, Józef. Thermal displacements in an elastic semi-space due to sudden heating of the boundary plane. Arch. Mech. Stos. **9** (1957), 395–416. (Polish and Russian summaries)

This paper contains a solution of a special problem in the theory of thermoelasticity. The problem is dynamical in the sense that it employs the dynamical equations of the theory of elasticity and includes the term $\partial T/\partial t$ in the equation governing the distribution of the temperature T; it does not however include in this equation the term proportional to the rate of change of the dilatation introduced by M. A. Biot [J. Appl. Phys. **27** (1956), 240–253; MR **17**, 1035] and therefore its solution cannot be described as a completely dynamical solution. The boundary conditions on the displacement field vector $u_i(x_1, x_2, x_3, t)$ are $u_i(x_1, x_2, 0, t)=0$, $u_i(x_1, x_2, x_3, 0)=0$, $\dot{u}_i(x_1, x_2, x_3, 0)=0$ so that they too are rather special and do not correspond to an obvious physical situation. The deformation of the solid is produced by a steady heat source of strength T^* situated at the point $(\xi_1^*, \xi_2^*, 0)$. The formal solution of the problem is obtained by introducing a "thermoelastic dynamic potential" function Φ with the property $u_i=\partial\Phi/\partial x_i$ and by the use of Fourier transforms; certain properties of retarded potentials are also employed. No numerical results are quoted. *I. N. Sneddon* (Glasgow).

Muki, Rokurō. Thermal stress in a semi-infinite solid and a thick plate under steady distribution of temperature. Proc. Fac. Engrg. Keio Univ. **9** (1956), 42–62 (1957).

This paper contains general solutions of the thermoelastic equations when the distribution of temperature is time-independent and the region considered is either a semi-infinite solid or a thick plate. In the case of a semi-infinite solid it is assumed that the plane boundary $z=0$ is free from stress and that the solid is deformed by the application of a temperature $\sum_m t_m(r)\cos m\theta$ to that boundary. In the case of a thick plate it is assumed that both faces are free from stress and that the temperature on each face is prescribed by a function of this general form. The method of solution is based on the method of Hankel transforms introduced by Harding and Sneddon [Proc. Cambridge Philos. Soc. **41** (1945), 16–26; MR **6**, 251] for the solution of axisymmetric problems and generalized by Muki [Proc. Fac. Engrg. Keio Univ. **6** (1952), 10–26] to the unsymmetric case. The author finds that each of the components of stress normal to the plane surfaces is zero. The use of the general solution is illustrated by the consideration of two special problems concerning thick plates. Detailed numerical calculations are illustrated graphically and the results compared with those derived for thin plates by J. E. Goldberg [J. Appl. Mech. **20** (1953), 257–260]. *I. N. Sneddon* (Glasgow).

Weiner, Jerome. An elastoplastic thermal-stress analysis of a free plate. J. Appl. Mech. **23** (1956), 395–402.

The thermal stresses in a free plate of elastoplastic material subject to a varying heat input over one face are determined. A heuristic solution is first found by suitable modifications of the elastic solution. It is then verified that the solution satisfies all the conditions of the appropriate uniqueness theorem.

See also: Partial Differential Equations: Lions. **Fluid Mechanics, Acoustics:** Stone.

Structure of Matter

Judd, B. R. A crystal field of icosahedral symmetry. Proc. Roy. Soc. London. Ser. A. **241** (1957), 122–131.

Bekannterweise besitzen die Ionen der seltenen Erden — da bei ihnen die 4*f*-Schale nicht komplett ist, ausserhalb der jedoch noch zwei weitere abgeschlossene Schalen (die 5*s*- und 5*p*-Schalen) liegen — im festen Zustande ein aus Linien bestehendes Absorptionsspektrum. Ein axialsymmetrisches Magnetfeld hebt die Entartung nach der magnetischen Quantenzahl bei freien Atomen oder Ionen vollständig auf, im erwähnten Falle der Ionen der seltenen Erden verursacht das meistens sehr symmetrische Kristallfeld nur eine teilweise Aufspaltung der Niveaus. Zur Berechnung von der hat H. Bethe [Ann. Physik. (5) **3** (1929), 133–208, S. 173], ausgehend aus der Gruppentheorie der Kristalle, eine gruppentheoretische Methode entwickelt. Der Verfasser zeigt erstens, dass sehr merkwürdige Verhältnisse auftreten, wenn das Kristallfeld eine ikosaedrische Symmetrie besitzt und berechnet dann, dass im Dysprosiummagnesiumnitrat das Feld tatsächlich mit grosser Annäherung diese Symmetrie besitzt. Die kleine Abweichung von der hat die Symmetrie C_{3v} (in der Bezeichnung von Schoenflies). Die Wirkung dieser Felder auf das Dy^{3+}-Ion, dessen Grundzustand ein ${}^6H_{15/2}$ Niveau ist, wird gruppentheoretisch berechnet. Dabei werden die Gruppencharaktere der doppelten ikosaedrischen und der ebenfalls doppelten C_{3v}-Gruppe und deren Ausreduktionen angegeben. Die Ergebnisse werden mit den experimentellen Resultaten von J. G. Park [Thesis, Oxford, 1957] verglichen, der aus der paramagnetischen Resonanz für die Landéschen g-Faktoren $g_{\|}=4{,}28$ und $g_{\perp}=8{,}92$ erhielt. Die theoretischen Resultate des Verfassers sind $g_{\|}=4{,}53$ und $g_{\perp}=9{,}07$. Ein sehr befriedigendes Ergebnis. *T. Neugebauer* (Budapest).

Price, P. J. The Lorenz number. IBM J. Res. Develop. **1** (1957), 147–157.

The theory of the Lorenz number of a conducting crystal is developed for the common models of the elec-

tron assembly. For the one-electron model it is shown that, provided scattering is elastic to an approximation which is examined, the Lorenz number is equal to the square fluctuation of the thermoelectric power. For the phenomenological band model an equivalent result is obtained. It is hypothesized that these results are special cases of a more general one. Some applications, including the theory of the bipolar anomaly for semiconductors, are discussed. (Author's abstract.) *P.-O. Löwdin.*

Lundqvist, Stig O. On the lattice vibrations in cubic ionic crystals. Ark. Fys. **12** (1957), 263–275.

This paper contains some further considerations connected with a previous paper [Ark. Fys. **9** (1955), 435–456] in which the long polar oscillations of a cubic ionic lattice were discussed on the basis of a potential energy, which is a first approximation of the expression for the cohesive energy in the Heitler-London treatment of ionic crystals. A somewhat more accurate estimate of the three-body potential is considered here, giving certain correction terms to the formula for the effective charge derived earlier. The limitations of the Heitler-London method in this connexion are discussed and a semi-empirical approach is suggested including also the polarisation. (Author's Abstract.) *P.-O. Löwdin* (Uppsala).

★**Seitz, Frederick; and Turnbull, David, editors. Solid state physics. Advances in research and applications. Vol. 1.** Academic Press Inc., New York, 1955. xii+469 pp. $10.00.

In this "Advances" series, both the theoretical and experimental aspects of solid state physics are to be covered in review papers, some of which will aim to be broad and elementary, others more advanced or specialized. The contents of the first volume are largely of theoretical character and, with one exception, are concerned with the electron theory of solids. In the first chapter "Methods of the One-Electron Theory of Solids", R. Reitz reviews the methods of calculating electron wave functions and electronic structures in solids based on the approximation that the total wave function can be represented as a combination of one-electron wave functions. After the derivation of the basic Fock and Hartree equations for these wave functions, the main part of the paper discusses methods for their solution. Most of the methods discussed involve solutions of the Bloch type (crystal orbitals) which extend through the whole crystal lattice, and there is a general discussion of crystal orbitals, introducing Brillouin zones, etc. A closely related problem in which the Schrödinger equation can be reduced to a Mathieu equation is then discussed. Bloch's original "tight binding method" of solution, in which the crystal orbital is constructed from a linear combination of atomic orbitals, is described, followed by its extension, the method of orthogonalized atomic orbitals. This is followed by the cellular method, pioneered by Wigner and Seitz, and its later developments, especially the orthogonalized plane wave method. Solutions of the Fock equation of the localized orbital type are discussed briefly in a section on the Heitler-London method. Finally, there is a brief section on exchange interaction in the one-electron approximation.

"The Quantum Defect Method" by F. S. Ham deals with a recently developed method (originated by Kuhn and van Vleck) for obtaining solutions of the Fock equation, in which knowledge of the one-electron Hamiltonian within the ion core is not required — it is assumed to be the same as for a valence electron in the free atom. While the method is presented in a form mainly applicable to the alkali metals, the possibility of its extension to other solids, in which core polarization, overlap and exchange interaction are important, is also discussed. It is shown how the solution of the wave equation can be written as a linear combination of two confluent hypergeometric functions outside the core. The coupling constant for the functions depends on a parameter obtained by drawing a smooth curve through the experimental values of the quantum defect when these are plotted on an energy scale. A mathematical justification of this procedure is given, as well as a somewhat different extrapolation that is better at low energies. The quantum defect method can then be used in conjunction with most of the usual procedures for calculating the energy bands of a solid, dealt with in Reitz's paper. Two final sections give quantum defect data for the alkali metals and their application to the calculation of lattice constants, cohesive energy and compressibility. An appendix discusses solutions of the hydrogenic radial wave equation

$$\frac{d^2U}{dr^2}+\left[-\frac{1}{n^2}+\frac{2}{r}-\frac{L(L+1)}{r^2}\right]U=0.$$

In "Qualitative Analysis of the Cohesion of Metals", E. P. Wigner and F. Seitz attempt to give a physical argument for predicting the cohesive properties of metals other than the alkali metals from a knowledge of atomic spectra, thus complementing Ham's paper.

In contrast to the papers on the one-electron theory, D. Pines' paper "Electron Interaction in Metals" reviews recent work on the many-body problem, mainly that by D. Bohm and himself. The Hamiltonian for the collective model is derived and reduced to a form in which it is evident that the long-range part of the Coulomb interaction between electrons can be described in terms of plasma oscillations. It is also shown that for phenomena not involving many atomic spacings, the use of the independent electron model is justified. The correlation energy of the electron gas is calculated and the influence of the lattice field considered. The theory is then applied to various metallic properties and previous difficulties with exchange interaction are resolved, since the spin induced correlations are now found to be effective only over a limited number of atomic spacings. Later sections deal with the excitation of plasma oscillations by fast charged particles and with the electron-lattice interaction.

Application of band theory is found in the chapter on "Valence Semiconductors, Germanium and Silicon" by H. Y. Fan, in which the role of impurities and imperfections must be included. There is a brief sketch of the relevant theory but the article is mainly devoted to the results of experimental work.

Finally, there is a chapter on "The Theory of Order-Disorder Transitions in Alloys" by T. Muto and Y. Tagaki. Both theoretical and experimental work is reviewed and it is shown that while the observations can be understood in a qualitative way from the present theories, many experiments are still needed. The principal theoretical problem is the statistical mechanical one of calculating the appropriate partition functions, for which various approximations have been made. These are discussed in some detail, from the early work of Bragg and Williams to the recent, more general methods exploited by Onsager. *M. S. Paterson.*

★**Seitz, Frederick; and Turnbull, David, editors. Solid state physics. Advances in research and applications. Vol. 2.** Academic Press Inc., New York, 1956. xii+468 pp. $10.00.

Except for the chapter on specific heats, the second volume in this series deals with subjects that have been advanced by techniques developed mainly since World War II. The first chapter is a general introduction to "Nuclear Magnetic Resonance" by G. E. Pake. After this property is described with the aid of rotating coordinates and the shape of the resonance peak is shown to depend on spin-spin and spin-lattice interactions, Bloch's phenomenological equations are discussed. Experimental methods for observing the resonance are then reviewed briefly. The later sections deal in more detail with sources of broadening of the resonance peak, with its shifts and splittings and with spin-lattice relaxation effects.

A related chapter by W. D. Knight on "Electron Paramagnetism and Nuclear Magnetic Resonance in Metals" deals only with shifts of the nuclear magnetic resonant frequency, especially the shift for an element between the metallic and combined states ("Knight shift"). The theory, involving paramagnetic susceptibility and probability density, is reviewed. Then there is a resumé of the experimental results available for metals and alloys.

In "Applications of Neutron Diffraction to Solid State Problems", C. G. Shull and E. O. Wollan deal with the fields of crystal structure and magnetic properties. In a brief introduction they describe the chief features of neutron diffraction and its merits in relation to X-ray diffraction. Thus its usefulness in locating hydrogen atoms and distinguishing between atoms of nearly identical X-ray scattering factors is illustrated in many examples of crystal structure studies. The section on magnetic studies describes examples of the use of neutron diffraction in elucidating the arrangement of spins and other ordering effects in ferromagnetic, antiferromagnetic and similar substances.

J. de Launay gives a comprehensive review of "The Theory of Specific Heats and Lattice Vibrations", beginning at an elementary level with the standard theories of Einstein, Debye, and Born and von Kármán. It is shown how comparison with experiment has necessitated successive refinement of the theories. The later sections describe developments of the Born and von Kármán theory, as applied to particular crystal lattices, and the calculation of the frequency spectra of thermal vibrations. The influences of the electron gas and departures from central forces are included in the discussion.

The last third of the volume is devoted to a review of "Displacements of Atoms during Irradiation" by F. Seitz and J. S. Koehler. While this is a subject of great practical importance in nuclear engineering, the authors "attempt to restrict attention to the fundamental and quantitative aspects of the subject". Thus they begin with the theory of collisions involving interaction of Coulomb fields, both in the nonrelativistic and relativistic ranges. The behaviour of knocked-on atoms and electron excitation during the passage of charged particles is discussed. The theory of temperature spikes (the regions of momentary high temperature resulting from transfer of energy from the incident particle to the lattice) and the possible stimulation of rate processes are shown to be important in understanding the influence of irradiation on order in alloys, strength properties, etc. The final sections review the many experimental observations on metals, valence crystals and ionic crystals in the light of the theories.

M. S. Paterson.

See also: Statistical Thermodynamics and Mechanics: Landau. **Optics, Electromagnetic Theory, Circuits:** Landauer. **Quantum Mechanics:** Froese. **Astronomy:** Mathur.

Fluid Mechanics, Acoustics

★**Tietjens, O. G. Fundamentals of hydro- and aeromechanics, based on lectures of L. Prandtl. Translated by L. Rosenhead.** Dover Publications, Inc., New York, 1957. xvi+270 pp. $1.85.

Like its companion volume "Applied hydro- and aeromechanics" [MR 19, 601], this volume is a re-issue of the English translation (1934) of a work first published in 1929. The work, which has since become a classic, consists of three parts: I. The statics of liquids and gases. II. Kinematics of liquids and gases. III. The dynamics of non-viscous fluids. These are subdivided into 15 chapters in all.

Moiseyev, N. N. Oscillations of bodies floating in a bounded reservoir. Dokl. Akad. Nauk SSSR (N.S.) **114** (1957), 1180–1183. (Russian)

The author announces theorems concerning the oscillations near equilibrium of a system of n bodies floating on the surface of a liquid in a bounded tank. The complicated equations governing the motions, when suitable reduced, have periodic solutions, the frequencies of which form an infinite sequence and $\omega_n^2 \to \infty$ as $n \to \infty$. A condition for stability, on which the formulation of a Cauchy problem subsequently depends, is given. The determination of the frequencies is reduced to finding the roots of a meromorphic function with simple poles.

C. D. Calsoyas (Livermore, Calif.).

Yih, Chia-Shun. Stream functions in three-dimensional flows. Houille Blanche **12** (1957), 445–450.

If (u, v, w) is the velocity of a homogeneous liquid at the point (x, y, z), the author calls stream functions an independent pair of solutions f and g of the differential equations $dx/u = dy/v = dz/w$ and obtains Clebsch's result that f and g may be so chosen that $(u, v, w) = \text{grad } f \times \text{grad } g$. If we are concerned with heterogeneous liquid (or with compressible fluid in steady motion),

$$(u, v, w) = (1/\rho)\text{grad } f \times \text{grad } g,$$

where ρ is the density. It is shown that the usual stream functions of twodimensional and axisymmetrical motions are included in the above definitions.

L. M. Milne-Thomson (Providence, R.I.).

Yih, Chia-Shun. Fonctions de courant dans les écoulements à trois dimensions. Houille Blanche **12** (1957), 439–444.

A French version of the paper reviewed above.

★**Giqueaux, Maurice. Mécanique des fluides théorique. Deuxième édition, revue et augmentée avec la collaboration de A. Oudart.** Libraire Polytechnique Ch. Béranger, Paris-Liége, 1957. xi+413 pp. 8800 francs.

This is an entirely theoretical treatise on fluid mechanics, with strong emphasis on aerodynamics. Thus the chapter on viscous flow is concerned mainly with boundary layers, and that on plane incompressible motion gives a thorough survey of airfoil theory (whereas such a classical subject as surface waves is not considered).

Following a clear exposition of kinematic and dynamic fundamentals, other chapters deal with Prandtl's wing theory, the vortex theory of propellers, free-streamline flow, compressibility, and wall interactions. The text is of about 1950 vintage so that, for example, transonic and hypersonic flows are not mentioned, nor the considerable theory of three-dimensional linearized supersonic flows.

M. D. Van Dyke (Los Altos, Calif.).

Jones, E. E. The forces on a thin aerofoil in slightly parabolic shear flow. Z. Angew. Math. Mech. **37** (1957), 362–370. (German, French and Russian summaries)

The lift, drag and couple acting on a thin Joukowski type aerofoil which is placed in a parabolic shear flow are calculated. The basic flow is taken to be of the form

$$u_0=-V\left(1+\frac{qy^2}{2h^2}\right),$$

where V is the velocity along the axis of symmetry and h is a reference length. The word "slightly" is inserted in the title because in the course of the analysis q^2 and higher powers are neglected. The differential equation governing the flow is solved in terms of Mathieu functions.

G. N. Lance (Southampton).

Ghosh, N. L. Potential flow with wake past a spherical obstacle. Proc. Nat. Inst. Sci. India. Part A. **23** (1957), 253–257.

Filippova, L. A. Unsteady motion of a viscous incompressible fluid in a narrow channel of constant width. Leningrad. Gos. Univ. Uč. Zap. **217**. Ser. Mat. Nauk. **31** (1957), 225–235. (Russian)

Liquid in time-dependent turbulent motion flows in a channel bounded by a fixed and a moving plane, the velocity of the latter being a given function of the time. The turbulent exchange coefficient $\mu_T=M\mu$ replaces the viscosity μ in the equation of motion and a slip velocity V_s is assumed. The author obtains formulae for M and for the ratio of V_s to the mean velocity. The method also applies to flow in a ring tube between two coaxial cylinders whose difference of radii is small compared with the radius of either, the outer tube being fixed and the inner one moving longitudinally. When $M=1$ and $V_s=0$ the results agree with those known for laminar flow.

L. M. Milne-Thomson (Providence, R.I.).

Nazarov, A. I. Oblique impact of a two-dimensional gas jet against a plane wall. Prikl. Mat. Meh. **21** (1957), 431–433. (Russian)

The problem is solved by applying Čapligin's method to the corresponding case for incompressible flow to give infinite series of hypergeometric functions. The solution agrees with the known solution when the impact is normal to the plate. *L. M. Milne-Thomson* (Providence, R.I.).

Musatov, V. V. On a case of motion of a free Kelvin wave. Prikl. Mat. Meh. **21** (1957), 347–352. (Russian)

An infinitely long canal of constant width l and variable depth rotates with a uniform angular velocity ω. The equation governing a simple-harmonic disturbance is

$$(*)\qquad \frac{\partial}{\partial x}\left(h\frac{\partial\zeta}{\partial x}\right)+\frac{\partial}{\partial y}\left(h\frac{\partial\zeta}{\partial y}\right)+\frac{\sigma^2-4\omega^2}{g}\zeta=\frac{2\omega}{i\sigma}\frac{\partial(\zeta,\,h)}{\partial(x,\,y)}$$

[cf. Lamb, Hydrodynamics, Cambridge, 1932, § 207] with the boundary condition

$$(**)\qquad \frac{\partial\zeta}{\partial y}-\frac{2\omega}{i\sigma}\frac{\partial\zeta}{\partial x}=0;\ y=-\tfrac{1}{2}l,\ y=\tfrac{1}{2}l,$$

where $\zeta(x, y)$ measures the deviation of the perturbed surface from the surface under rotational equilibrium. Assume that the depth of the canal varies linearly with the width. Put $h(y)=h_0H(y)$, h_0 a parameter, and $\zeta(x, y)=e^{ikx}Y(y)$; then (*) becomes

$$(***)\qquad \frac{d}{dy}\left(H\frac{dY}{dy}\right)+\left(\kappa^2-k^2H-\frac{2\omega}{\sigma}k\frac{dH}{dy}\right)Y=0;\qquad \left(\kappa^2=\frac{\sigma^2-4\omega^2}{gh_0}\right)$$

and (**) becomes

$$(****)\qquad \frac{dY}{dy}-\frac{2\omega}{\sigma}kY=0;\ y=-\tfrac{1}{2}l,\ y=\tfrac{1}{2}l.$$

Further transformations reduce (***) to the normal form

$$\frac{d^2S}{d\tau^2}+[K^2\mu-q(\tau)]S=0$$

with appropriate boundary conditions. The asymptotic solution of this equation gives the relation between the frequency σ and the wave number k

$$\sigma^2=4\omega^2+gk^2\frac{h_1+\sqrt{(h_1h_2)}+h_2}{3}+\frac{3g(h_2-h_1)^2}{16l^2\sqrt{(h_1h_2)}}+\frac{\pi^2g}{l^2}\left(\frac{\sqrt{h_1}+\sqrt{h_2}}{2}\right)^2(n^2+O(n^{-1})),$$

where h_1 is the canal depth at $y=-\frac{1}{2}l$ and h_2 at $y=\frac{1}{2}l$. This equation includes Lagrange's formula $\sigma^2=ghk^2$ ($\omega=0$, $l\to\infty$, $h_1=h_2$).

C. D. Calsoyas.

Crane, L. J. The laminar and turbulent mixing of jets of compressible fluid. II. The mixing of two semi-infinite streams. J. Fluid Mech. **3** (1957), 81–92.

The author applies the methods developed in part I [same J. **2** (1957), 449–455; MR **19**, 602] to the mixing of two parallel compressible streams for both laminar and turbulent flows. The method used consists in developing the stream function in a double series of powers of two parameters involving, respectively, the square of Mach number and the temperature difference. Solutions are found for terms up to the second order in the series when the streams do not differ too greatly in velocity and temperature. When one of the streams is at rest, the method is no longer sufficiently accurate and numerical solutions are given.

In the case of turbulent jet, the results show that the effect of "change of scale" and the effect of the perturbation-terms due to Mach number are approximately equal and opposite, leaving the velocity profile sensibly unchanged from that of the incompressible flow.

Y. H. Kuo (Peking).

Squire, H. B. The motion of a simple wedge along the water surface. Proc. Roy. Soc. London. Ser. A. **243** (1957), 48–64.

An investigation is made of the two-dimensional flow of a stream past a wedge partially immersed in water. The wedge is replaced by a moving pressure distribution chosen to satisfy some rather arbitrary conditions. Formulae are obtained for the depth of immersion and resistance. It is found that the immersion increases to a maximum and then becomes small as the speed increases.

The resistance follows the same pattern with the maximum at a speed slightly greater than that of maximum immersion. The numerical details are found in the paper reviewed below. *R. C. MacCamy* (Pittsburgh, Pa.).

Miller, G. F. On certain integrals occurring in a hydrodynamical problem. Proc. Roy. Soc. London. Ser. A. 243 (1957), 65–77.

The integrals,

$$C_n(x, k)=p\int_0^\infty \frac{J_n(t)\cos xt}{t-k}\,dt;\quad S_n=p\int_0^\infty \frac{J_n(t)\sin xt}{t-k}\,dt$$

occurring in the paper reviewed above are studied. Extensive tables are given as are asymptotic expressions for large k. The results are used in the paper of Squire. *R. C. MacCamy* (Pittsburgh, Pa.).

Biesel, F.; et Le Méhauté, B. Mouvements de seiches à trois dimensions. Houille Blanche 12 (1957), 430–438.

Bhattacharya, R. N. Wave resistance in deep water due to the accelerated motion of a pressure system. Proc. Nat. Inst. Sci. India. Part A. 23 (1957), 191–198.

Stewartson, K. On almost rigid rotations. J. Fluid Mech. 3 (1957), 17–26.

Proudman [same J. 1 (1956), 505–516; MR 18, 618] investigated the motion of an incompressible viscous fluid confined between two concentric spheres which rotate about a common diameter with angular velocities Ω and $\Omega(1+\varepsilon)$ where ε is small. Proudman left some unsolved questions concerning shear layers. The present paper is concerned with a study of the flow between two coaxial rotating discs, each having an arbitrary small angular velocity superposed on a finite constant angular velocity, and throws some light on Proudman's questions.

A similar problem of two coaxial parallel discs rotating with the same angular velocity but enclosed in a coaxial cylinder rotating with a slightly different angular velocity is also discussed. *A. E. Green* (Newcastle-upon-Tyne).

Hanin, Meir. Propagation of an aperiodic wave in a compressible viscous medium. J. Math. Phys. 36 (1957), 234–249.

An investigation of the propagation of the plane wave due to a Dirac-pulse disturbance in such a medium leads via Heaviside transforms to a consideration of the basic solution

$$2\pi i U(x, t)=\int_{c-i\infty}^{c+i\infty} \exp[t\sigma-x\sigma(1+\sigma)^{-\frac{1}{2}}]d\sigma$$

in non-dimensional variables. Various expansions and real integral representations valid in suitable ranges of x and t are derived and used to portray the progress of the wave. The dispersive effect of viscosity is clearly shown. *H. C. Levey* (Melbourne).

Hanin, Meir. Mathematical results related to a problem in wave propagation. J. Math. Phys. 36 (1957), 250–260.

Summation formulae are derived for generalisations of Neumann series which occur in the paper reviewed above. A typical result is that

$$\sum_{n=0}^{\infty} \frac{y^n}{(\nu+\frac{1}{2}n)!}\,J_{\mu+n}(x)=2^{2\nu}x^\mu\int_0^\infty E_{2\nu}(\sigma)z^{-\mu}J_\mu(z)d\sigma$$

where

$$z^2=x(x-4y\sigma),$$

$$E_\lambda(\sigma)=2\pi^{-\frac{1}{2}}2^{-\frac{1}{2}\lambda}e^{-\frac{1}{2}\sigma^2}D_{-\lambda}(\sigma 2^{\frac{1}{2}}).$$

The last section is devoted to the discussion of a class of functions satisfying a special ordinary third order differential equation. *H. C. Levey* (Melbourne).

★ **Comolet, Raymond. Écoulement d'un fluide entre deux plans parallèles. Contribution à l'étude des butées d'air.** Publ. Sci. Tech. Ministère de l'Air, Paris, no. 334, 1957. iv+68 pp. (15 plates). 1100 francs.

With simplifying assumptions, the laminar two-dimensional or radial flow of a viscous perfect gas between two parallel walls is treated. All the variables except the speed are assumed to have no variation normal to the walls, and the gas is assumed to behave polytropically. The equations for conservation of mass and momentum (with effectively a boundary layer type assumption) then yield a relation between mass flow and pressure gradient. Experimental work for the case of radial flow is claimed to verify this relation provided the polytropic index is taken to be unity, that is, the gas behaves isothermally. It should be noted that the energy equation is not used and apparently would not be satisfied, and that temperature boundary conditions on the walls appear to be disregarded throughout, both in the theoretical and experimental work. *H. C. Levey* (Melbourne).

Herzig, Howard Z.; and Hansen, Arthur G. On possible similarity solutions for three-dimensional incompressible laminar boundary layers. II. Similarity with respect to stationary polar coordinates. NACA Tech. Note no. 3832 (1956), 16 pp.

Nachdem die gleichen Verfasser bereits früher ermittelt haben [NACA Tech. Note no. 3768 (1956)], welche in rechtwinkligen Koordinaten darstellbaren Strömungen ähnliche Lösungen der Grenzschichtgleichungen ergeben, werden diese Untersuchungen jetzt auf Strömungen in Polarkoordinaten (r, θ, y) ausgedehnt. Es zeigt sich, dass es hier nur einen Strömungstyp gibt, der zu ähnlichen Lösungen der Grenzschichtgleichungen führt,

$$U=ar^n e^{m\theta},\quad W=br^n e^{m\theta},$$

wobei mit U die Geschwindigkeitskomponente der Aussenströmung in Richtung von r und mit W die Komponente senkrecht dazu bezeichnet ist; a, b, n, m sind beliebige Konstanten. Die zu diesen Strömungen gehörigen Grenzschichten lassen sich nicht unmittelbar auf praktische Fälle übertragen, da der Beginn der Grenzschichten entweder im Nullpunkt ($n<1$) oder im Unendlichen ($n>1$) liegen muss. Die weiteren Eigenschaften dieser Grenzschichten sind in einer Tabelle zusammengestellt. Gleichzeitig sind Literaturstellen angegeben, wo bereits für ausgezeichnete Werte der Konstanten a, b, n, m die Grenzschicht numerisch ermittelt worden ist.

L. Speidel (Mülheim).

Herzig, Howard Z.; and Hansen, Arthur G. On possible similarity solutions for three-dimensional incompressible laminar boundary layers. III. Similarity with respect to stationary polar coordinates for small angle variation. NACA Tech. Note no. 3890 (1957), 36 pp.

Die von den gleichen Verfassern bereits durchgeführten Untersuchungen [siehe der oben referierten Arbeit] hatten Strömungsformen ergeben, die nicht unmittelbar auf praktische Fälle übertragen werden konnten, da die zugehörigen Grenzschichten entweder im Nullpunkt oder

im Unendlichen beginnen mussten. Um zu praktisch verwendbaren Lösungen zu kommen, wird die Untersuchung auf solche Strömungen ausgedehnt, die nur in einem kleinen Bereich von θ (Winkel der Polarkoordinaten) ähnliche Lösungen der Grenzschichtgleichungen ergeben. Man erhält jetzt eine Klasse von Strömungen mit Kreisen oder Spiralen als Strömlinie, deren Grenzschichten bei $\theta=0$ beginnen. Diese Strömungsformen sind mit den wichtigsten Grenzschichteigenschaften in einer Tabelle zusammengestellt. An der Strömung durch einen kreisförmigen rechteckigen Kanal wird die Übereinstimmung zwischen Rechnung und Messung gezeigt. *L. Speidel.*

Wadhwa, Y. D. Boundary layer growth on a spinning body: Accelerated motion. Phil. Mag. (8) **3** (1958), 152–158.

This paper extends to the case of accelerated motion the calculation of Illingworth [Phil. Mag. (7) **45** (1954), 1–8; MR **15**, 477] on spinning bodies of revolution started impulsively from rest. Only laminar boundary-layer flow is considered, of course, and the fluid is incompressible. The general results are obtained in the form of power series in the time variable, and are applied to the special case of a body uniformly accelerated from rest. Numerical results are given for spheres with various dimensionless rates of rotation. It is found that "separation", i.e., the occurrence of vanishing shear at the body surface, occurs earlier when there is rotation; this agrees with Illingworth. *W. R. Sears* (Ithaca, N.Y.).

Cooke, J. C. The flow of fluid along cylinders. Quart. J. Mech. Appl. Math. **10** (1957), 312–321.

The paper considers the laminar incompressible boundary layer flow over a cylinder of arbitrary cross-section (continuous radius of curvature), when the external flow is uniform and parallel to the cylinder generators. Zero cross-flow is assumed for the boundary layer flow and a solution is obtained in terms of a power series expansion in a similarity variable. It is shown that the first three terms of the expansion are the same as those for a circular cylinder, and that only further terms depend upon the radius of curvature and its derivatives. A von-Karman-Pohlhausen integral procedure is also carried out for the problem. As in the previous method the first three terms are the same as for a circular cylinder. Some calculations are carried out for the drag on an elliptical cylinder. It is indicated that the drag is less than that of a circular cylinder of the same perimeter. *P. Chiarulli.*

Sanyal, Lakshmi. Two-dimensional boundary layer flow along a wall in a converging channel with curved boundaries. Bull. Calcutta Math. Soc. **47** (1955), 129–133.

Rościszewski, Jan. An approximate solution of laminar boundary layer equations by means of polynomials. Arch. Mech. Stos. **9** (1957), 381–393. (Polish and Russian summaries)

Curle, N.; and Skan, S. W. Approximate methods for predicting separation properties of laminar boundary layers. Aero. Quart. **8** (1957), 257–268.

Roy, Maurice. Nappe en cornet et tourbillons d'apex de l'aile en delta. C. R. Acad. Sci. Paris **244** (1957), 1105–1108.

In this note, on vortex sheets springing from the apex of a delta wing, the author continues the interesting and important researches which he initiated in a previous note on the subject. [same C. R. **234** (1952), 2501–2504]. These sheets are curled up into a hornlike structure and the author gives a linearised theory which seems to require a distribution of sources and sinks off the wing. *G. Temple* (Oxford).

Sanyal, Lakshmi. A note on Jeffery's exact solution of steady two-dimensional motion of a viscous liquid. Bull. Calcutta Math. Soc. **47** (1955), 125–127.

Janssens, Paul. Sur le rôle des corrélations en turbulence homogène. Acad. Roy. Belg. Cl. Sci. Mém. Coll. in 8° **30** (1957), no. 5, 112 pp.

Ce mémoire constitue une revue d'ensemble des recherches sur la cinématique et la dynamique des corrélations en turbulence homogène. Le chapitre premier est une introduction historique, où sont rappelés les travaux de Boussinesq, Reynolds, Taylor, Karman et Howarth, Kolmogoroff, Heisenberg, ainsi que ceux des dernières années.

Dans le chapitre II (cinématique de la turbulence) sont définies les diverses grandeurs statistiques: lois de probabilité, moyennes, corrélations. Un paragraphe est consacré à la signification des valeurs moyennes: la valeur moyenne de $f(t)$ est une fonction $\langle f\rangle$, déduite de f par une opération linéaire, et qui satisfait en outre aux conditions suivantes:

$$\langle\langle f\rangle, g\rangle=\langle f\rangle\langle g\rangle,\ \langle \lim_{n\to\infty} f_n\rangle=\lim_{n\to\infty}\langle f_n\rangle.$$

Les moyennes, spatiales ou temporelles, au sens de Reynolds, ne peuvent vérifier rigoureusement ces conditions que dans le cas de la turbulence homogène ou stationnaire. Les moyennes stochastiques sont plus correctes, mais leur confrontation avec l'expérience est plus difficile. Dans le cas d'un fluide compressible de densité ρ, on a intérêt à introduire les moyennes pondérées de Van Mieghen, définies par

$$\{z\}=\frac{\langle\rho z\rangle}{\langle\rho\rangle}.$$

On introduit dans la théorie de la turbulence divers tenseurs de corrélation spatiale relatifs à la vitesse en deux points:

$$R_{ij}=\langle u_iu_j'\rangle,\ T_{ijk}=\langle u_iu_ju_k'\rangle,\ D_{ijkl}=\langle u_iu_ju_ku_l'\rangle,$$
$$Q_{ijkl}=\langle u_iu_ju_k'u_l'\rangle.$$

On introduit aussi des tenseurs de corrélation où entre le tourbillon de la vitesse, puis la pression. Diverses relations sont satisfaites par ces tenseurs. En turbulence homogène, R_{ij} contient 19 fonctions de 3 variables, même si le fluide est compressible. Leurs développements suivant les puissances de la distance r des deux points permettent de discuter la décroissance de la turbulence, par l'intermédiaire de longueurs de corrélation que l'expérience permet de mesurer.

L'Auteur examine les réductions que provoquent les symétries: axisymétrie, isotropie, et donne les expressions réduites des tenseurs de corrélation usuels entre la vitesse, la pression, et les dérivées spatiales de la vitesse. Après quelques indications sur l'analyse de Fourier de la turbulence, il rappelle les principales hypothèses physiques ou mathématiques qui ont été introduites pour achever de calculer les corrélations: quasi-normalité, isotropie locale et analyse dimensionnelle (Kolmogoroff).

Le chapitre III (dynamique des corrélations) rappelle d'abord l'équation de Karman-Howarth, qui relie R_{ij}, T_{ijk}

et $\overline{pu_j'}$ en turbulence homogène. De cette équation découlent des équations locales relatives à l'évolution de l'énergie cinétique de la turbulence. Elles donnent des renseignements sur l'évolution de la turbulence vers l'isotropie. La turbulence admet divers invariants intégraux, dont le plus connu est celui de Loitsiansky. Lorsque la vitesse moyenne U_i est uniforme, mais décroît en fonction du temps, il est possible de généraliser l'équation de Karman-Howarth.

L'Auteur passe ensuite en revue certaines solutions de l'équation de Karman-Howarth: corrélations triples négligeables, solutions affines de Batchelor, puis les diverses hypothèses (Batchelor, Heisenberg, von Weizsäcker, von Karman, Kovasznay, Davies) qui ont permis d'expliciter la forme de la fonction spectrale $F(k, t)$.

Le chapitre IV est consacré à l'influence de la compressibilité sur l'évolution de la turbulence homogène et isotrope. A l'équation de l'énergie cinétique, fournie par les équations de Navier, il faut ajouter une équation thermodynamique. On peut généraliser les formules des corrélations. L'équation de dissipation de l'énergie est enfin établie et sommairement discutée.

Le mémoire se termine par une importante bibliographie. *J. Bass* (Paris).

Ghosh, K. M. On localized axisymmetric turbulence. Proc. Nat. Inst. Sci. India. Part A. **23** (1957), 341–353.

Batchelor et Chandrasekhar ont étudié la turbulence axisymétrique homogène non stationnaire, dans laquelle les divers tenseurs de corrélation de la vitesse sont invariants aux rotations autour des axes de direction λ et aux symétries par rapport aux plans contenant λ ou perpendiculaires à λ. Bass a étudié la turbulence cylindrique stationnaire, analogue à la précédente, mais non homogène dans la direction λ. Ces deux problèmes s'appliquent à un écoulement dans un domaine limité, de forme générale cylindrique.

L'auteur se propose d'étudier la turbulence axisymétrique stationnaire autour d'un axe choisi, homogène longitudinalement, mais non transversalement, le fluide pouvant alors occuper tout l'espace.

Se limitant aux corrélations spatiales $F_{ij}=\overline{u_iu_j'}$ en deux points P et Q, il en donne la forme réduite, qui dépend de 10 fonctions de 4 produits scalaires. L'incompressibilité établit entre ces fonctions diverses relations assez compliquées. La forme des développements limités des F_{ij} pour des petites valeurs de la distance PQ est ensuite donnée. F_{ij} n'est pas un tenseur symétrique.

L'étude des surfaces et des courbes d'iso-corrélation suggère des méthodes expérimentales pour mesurer les constantes dont dépendent localement les F_{ij} et comparer les hypothèses faites à l'expérience. *J. Bass* (Paris).

Lièvre, Roger. Étude théorique de la diffusion saline turbulente en conduite en régime permanent. C. R. Acad. Sci. Paris **244** (1957), 1453–1455.

Bagdoev, A. G. Penetration of pressure to the bottom of a compressible inhomogeneous fluid. Vestnik Moskov. Univ. Ser. Mat. Meh. Astr. Fiz. Him. **12** (1957), no. 2, 45–50. (Russian)

Compressible fluid occupying a lower half space is stratified in layers and a given axisymmetric pressure distribution is applied at the free surface. The author finds expressions for the pressure distribution at depth z and time t in the particular cases of (i) two layers; (ii) an exponential law of change of density with depth.

L. M. Milne-Thomson (Providence, R.I.).

Belova, A. V. Force exerted by a subsonic stream of gas on a profile. Leningrad. Gos. Univ. Uč. Zap. **217**. Ser. Mat. Nauk **31** (1957), 90–123. (Russian)

A plane subsonic stream of gas in irrotational motion is disturbed by a profile. The author applies the method of Hristianovič [Trudy Central Aero-gidrodinam. Inst. no. **481** (1940); MR **7**, 39] to determine the circulation Γ and the moment M in the forms

$$\Gamma=[\Gamma_0+\Gamma_1(\tfrac{1}{2}M_\infty^2)+\cdots+\Gamma_k(\tfrac{1}{2}M_\infty^2)^k+\cdots](1-M_\infty^2)^{-\frac{1}{2}},$$

$$M=M_0+M_1(M_\infty^2)+\cdots+M_k(M_\infty^{2k})+\cdots,$$

where M_∞ is the Mach number of the undisturbed stream. The determination of the coefficients is discussed in detail and Γ_1 is found for a plate, a circular arc, and a Joukowski profile, and in the latter case expressions for M_1, M_2, M_3, M_4 are also found. *L. M. Milne-Thomson.*

Belova, A. V. Moment of the profile of a wing at high subsonic speeds. Leningrad. Gos. Univ. Uč. Zap. **217**. Ser. Mat. Nauk **31** (1957), 124–143. (Russian)

A formula is derived for the moment of the profile of a wing placed in a steady, two-dimensional, irrotational, compressible flow. The result, which is written in the form

$$(*)\qquad M=\frac{\rho_\infty V_\infty^2}{b^2}\left[\pi\beta_1\left(1+\frac{1}{\sqrt{(1-M_\infty^2)}}\right)-\alpha_0\gamma\right],$$

generalizes a well-known formula applicable to incompressible flow [see, e.g., L. M. Milne-Thomson, Theoretical hydrodynamics, 2nd ed., Macmillan, New York, 1950, p. 188, eq. (8); MR **17**, 796]. In (*), ρ_∞, V_∞, M_∞ are the free stream density, velocity, and Mach number; b is the scale factor associated with the mapping function

$$(**)\qquad z=\frac{1}{b}\left(\zeta+a_0+\frac{a_1}{\zeta}+\frac{a_2}{\zeta^2}+\cdots\right),$$

which maps the region exterior to the circle $|\zeta|=1$ conformally onto the region exterior to the wing profile in the z-plane; $\alpha_0=\mathrm{Re}\,a_0$; $\beta_1=\mathrm{Im}\,a_1$; and $\gamma=b\Gamma/V_\infty$, the circulation Γ being given as $[1-M_\infty^2]^{-\frac{1}{2}}$ times a power series in $(\frac{1}{2}M_\infty^2)$.

The derivation of (*) represents a prodigious amount of work, depending in part on a previous paper by the author [see the paper reviewed above]. The accuracy afforded by (*) reflects inclusion of terms of order $1/|\zeta|$ in the analysis. *J. F. Heyda* (Cincinnati, Ohio).

Gispert, Hans-Günter. Numerische Behandlung eines 2-dimensionalen Variationsproblems aus der Gasdynamik. Wiss. Z. Martin-Luther-Univ. Halle-Wittenberg. Math.-Nat. Reihe **6** (1956/57), 209–221.

The author investigates the plane, steady, irrotational subsonic flow about a circle or ellipse. The potential Φ minimizes the integral

$$\iint\left[c_0^2-\frac{\kappa-1}{2}(\Phi_x^2+\Phi_y^2-v_0^2)\right]^{2\kappa/\kappa-1}dxdy,$$

where c_0, v_0, κ are sound speed, speed at ∞ and specific heat ratio. G. Braun [Ann. Physik. (5) **15** (1932), 645–676] attacked the problem of finding Φ by assuming the difference from incompressible flow to be small. The problem has also been treated by the Janzen-Rayleigh method by I. Imai [Proc. Phys.-Math. Soc. Japan (3) **23** (1941), 180–193] and E. Lamla [Jbuch. Deutsch. Luftfahrtforschung **1939**, 565–578]. The author finds the approximate solution for a circle by minimizing the integral with respect to functions of the form $\Phi+f_1(s)\cos\theta+$

$f_3(s)\cos 3\theta + f_5(s)\cos 5\theta$, where Φ_1 is the incompressible flow, θ is the radial angle and s is the reciprocal of the radius. The functions f_1, f_3, f_5 satisfy three second order non-linear differential equations which are solved numerically. The results agree well with those of the Janzen-Rayleigh method for free stream Mach numbers of 0.35 and 0.40. The same method may be applied to other profiles and is carried out for an ellipse. *C. S. Morawetz.*

Morioka, Shigeki. Transonic flow with a detached bow wave past a wedge between two parallel plane walls. J. Aero. Sci. **24** (1957), 831–837.

The author applies the methods of Vincenti and Wagoner [NACA Rep. no. **1095** (1952); MR **18**, 845] to determine the flow described in the title in order to study the effect of wind tunnel walls on the bow wave and on pressure distribution. The mathematical problem reduces to solving the Tricomi problem with a partly free elliptic boundary. It differs from the problem of Vincenti and Wagoner in that the free boundary (the shock) is a slit in the hodograph plane. When the tunnel width decreases the bow wave moves forward and the pressure coefficients over the profile increase uniformly. For some range of tunnel widths the solution is not unique and in fact breaks down for a limiting value of the tunnel width. Results are compared with Vincenti and Wagoner (wall at infinite distance) and Marschner (shock at infinite distance). The mean mass flow at the narrowest passage in the limiting case is clearly larger than that for the subsonic choked condition (Marschner's case).

C. S. Morawetz (New York, N.Y.).

Woods, L. C. On the lifting aerofoil in a wind tunnel with porous walls. Proc. Roy. Soc. London. Ser. A. **242** (1957), 341–354.

This is the second of two papers by the author dealing with the subsonic two-dimensional flow past an aerofoil at rest in a wind tunnel with porous walls. The first [same Proc. **233** (1955), 74–90; MR **17**, 550] concerned the blockage effect at the model due to the tunnel walls and in the present paper the wind tunnel corrections to the lift and moment acting on the aerofoil are derived. The tunnel walls are taken to be porous over a finite length and solid elsewhere. The theory allows for the possibility of different pressures outside of the two porous sections and it is suggested that this facility may be useful in the elimination or reduction of interference effects. *D. G. Drake.*

Eckhaus, W. A method for the asymptotic expansion of the integral equation of a lifting surface at near-sonic speeds. Nat. Luchtvaartlab. Amsterdam NLL-TN F. **200** (1957), 15 pp.

The author considers the integral equation of lifting surface theory in subsonic compressible flow, that is,

$$(*)\quad w(x,y)=\iint_R \phi(x',y')\left\{1+\frac{x-x'}{[(x-x')^2+\alpha^2(y-y')^2]^{\frac{1}{2}}}\right\}\frac{dx'dy'}{(y-y')^2},$$

R being a region in the x', y' plane. The parameter α is the product of the Mach number and the slenderness ratio of the airfoil, w is given, and ϕ is to be determined. By a process of splitting the region of integration into a portion near the singularity $x=x'$ and a remainder, equation (*) is rewritten,

$$(**)\qquad w=T_1(\phi)+\alpha^2 T_2(\phi)+O(\alpha^3),$$

T_1 and T_2 being integral operators independent of α. Some rather vague remarks are made concerning the validity of the approximation near the trailing edge.

R. C. MacCamy (Pittsburgh, Pa.).

Šmyglevskiĭ, Yu. D. Some variational problems of gas dynamics of axi-symmetric supersonic flows. Prikl. Mat. Meh. **21** (1957), 195–206. (Russian)

The axisymmetric contour having minimum wave drag in purely supersonic flow, connecting a given nose with a given downstream point, is sought. Following Nikolsky (unpublished), the full irrotational equations of gas dynamics are applied to the curvilinear triangle bounded by the unknown contour and by the known downstream and unknown upstream characteristics from its extremities. The resulting variational problem is found to be degenerate, the unknown functions being subjected to one too many conditions. This difficulty is removed by observing on physical grounds that the contour will join the nose at an outside corner, from which will spring a curvilinear expansion fan. The extremal relations apply only downstream of the fan, and the extra degree of freedom involved in the unknown width of the fan renders the problem soluble. It is verified that the extremum of drag is a minimum. Extensions are outlined to flow inside a nozzle and to flow with vorticity. Numerical results are given for contours that follow a 35° conical nose and a semi-infinite cylinder at Mach numbers of 2 and 3. *M. D. Van Dyke* (Los Altos, Calif.).

Šmyglevskiĭ, Yu. D. A variational problem in the gas dynamics of axially symmetrical supersonic flows. Dokl. Akad. Nauk SSSR (N.S.) **113** (1957), 520–522. (Russian)

A brief preliminary report of work that was published in full in the article reviewed above. *M. D. Van Dyke.*

Ferri, Antonio; Clarke, Joseph H.; and Ting, Lu. Favorable interference in lifting systems in supersonic flow. J. Aero. Sci. **24** (1957), 791–804.

The paper shows how certain combinations of lifting surfaces may be obtained which reduce the drag of the configuration. The particular combinations considered include (1) a slender body under a lifting wing, (2) wedges in the vertical plane of symmetry of a lifting wing, and (3) a pylon under a lifting wing. In each case, it is found that when the two parts of the combination are placed in certain lifting positions which are stated, the drag is minimized for a specific Mach number. The theory of the present paper is an extension of the area rule for drag reduction which is applicable to non-lifting configurations.

G. N. Lance (Southampton).

Napolitano, Luigi G.; and Ferri, Antonio. The axisymmetric supersonic flow near the nose of a pointed body of revolution. J. Aero. Sci. **24** (1957), 900–904.

An approximate method of solution for the supersonic flow past axisymmetric bodies is given. The flow is represented as the superposition of a basic non-linear flow field (appropriate to the leading edge tangent-cone in this application), and small perturbation fields which are treated by the method of linearized characteristics. The method is claimed to be rapid and accurate, and results obtained for the RM-10 body agree well with those found by other means. *H. C. Levey* (Melbourne).

★ **Kubota, Toshi. Inviscid hypersonic flow over blunt-nosed slender bodies.** Heat transfer and fluid mechanics institute, held at California Institute of Technology, Pasadena, Calif., June, 1957, pp. 193–210. Stanford University Press, Stanford, Calif. $7.50.

A solution of the hypersonic small disturbance equations is obtained for non-lifting, symmetric blunt-nosed bodies by using the similarity property of the flow, i.e., in the limiting case of infinite Mach number, the ratio of the transversal component of the velocity, the pressure and the density at the distance r from the axis to those immediately behind the shock wave $r=R(x)$ are functions only of r/R for body shapes $r_b \sim x^m$ provided $1 \geqq m \geqq 2/3$ for two-dimensional flows and $1 \geqq m \geqq \frac{1}{2}$ for axially symmetric flows. For large but finite Mach numbers a second approximation of the form

$$Q(x, r)=g(x)[f_0(r/x^m)+f_1(r/x^m)x^{2(1-m)}/M_\infty{}^2\delta^2]$$

is obtained.

These results were compared with the experimental study of the flow around such bodies conducted at Mach number 7.7 in the GALCIT Hypersonic Wind Tunnel, and the surface pressure distribution and shock shape show good agreement with the theoretical prediction. (Author's summary.) *W. Littman* (Berkeley, Calif.).

Kogan, M. N. Properties of spatial supersonic flows. Prikl. Mat. Meh. **20** (1956), 661–662. (Russian)

Gadd, G. E. A theoretical investigation of laminar separation in supersonic flow. J. Aero. Sci. **24** (1957), 759–771, 784.

Maslen, S. H.; and Moeckel, W. E. Inviscid hypersonic flow past blunt bodies. J. Aero. Sci. **24** (1957), 683–693.

★ **Austern, N.; and Percus, J. K. Propagation of strong blast in an atmosphere of varying density.** AEC Computing and Applied Mathematics Center, Institute of Mathematical Sciences, New York University, 1957. 35 pp.

The following problem is analyzed by various methods. A strong blast occurs at a point on the earth and propagates into the air, the density of which is assumed linear in height. It is assumed that the blast remains sufficiently strong so that the effects of gravity may be neglected. The solution is first found as a perturbation about Taylor's blast solution. For example, the correction to the distance R from the origin of the blast to the shock, as a function of time and vertical angle θ, is proportional to $R_0{}^2 \cos\theta$, where $R_0 \propto t^{2/5}$ is given by Taylor's solution. Secondly, it is shown that the snow plow model ($\gamma=1$ gas) yields comparatively good results and with some fairly natural corrections even better results. *C. S. Morawetz.*

Tsugé, Shun-ichi. On a theory of shock waves in locally supersonic zone and a new boundary-value problem for Tricomi's equation. J. Phys. Soc. Japan **12** (1957), 1412–1419.

L'auteur étudie l'écoulement permanent transsonique d'un fluide compressible autour d'un profil à deux dimensions. L'approximation utilisée est celle qui conduit à l'équation de Tricomi pour la fonction de courant. Des relations relatives à l'onde de choc sont établies, qui permettent de conclure que cette onde est située entre les deux branches de la ligne limite de l'écoulement continu. *H. Cabannes* (Marseille).

Payne, R. B. A numerical method for a converging cylindrical shock. J. Fluid Mech. **2** (1957), 185–200.

This paper describes a numerical method for computing compressible flows with cylindrical symmetry, with shocks present. The method used is a modification of one given by the reviewer [Comm. Pure Appl. Math. **7** (1954), 159–193; MR **16**, 524], which converts a system of conservation laws into difference equations in conservation form. The variables used are mass and momentum per unit cylindrical shell thickness, and radius times radial momentum. The momentum equation contains a term not in conservation form, which moreover becomes singular at the origin. The author has devised a special way of treating this additional term. Results of calculations of converging shock waves are presented; they show good agreement with a formula given by Chisnell. There is also a discussion of the limitations of this method.

This paper is recommended to the attention of all who are interested in cylindrical and spherical shocks. *P. D. Lax* (New York, N.Y.).

Paul, David I. Acoustical radiation from a point source in the presence of two media. J. Acoust. Soc. Amer. **29** (1957), 1102–1109.

Following the method applied by Baños and Wesley [Scripps Inst. Oceanog., Univ. Calif., part I, reference 53–33 (1953) and Part II, reference 54–31 (1954) to the corresponding electromagnetic problem, the author obtains asymptotic expressions for the velocity potential due to a point source in the presence of two semi-infinite (porous or nonporous) media separated by a plane interface. Assuming that the point source is situated near the interface and that $k_1R \gg 1$, where k_1 are the wavenumbers for the two media and R the distance to the observation point, the asymptotic expansions, carried out to three terms in inverse powers of k_1R, are valid either near the interface or near the vertical axis through the source. The expansion near the interface takes on different forms depending on whether $k_2R\delta^2(1-n^2)$ is large or small, where $n=k_1/k_2$ and δ is the ratio of the densities of the two media.

Calculations have been carried out for the amplitude of the velocity potential along an interface between air and sea water, and between air and USG Quietone. *L. B. Felsen* (Brooklyn, N.Y.).

Green, A. E.; and Rivlin, R. S. Steady flow of non-Newtonian fluids through tubes. Quart. Appl. Math. **14** (1956), 299–308.

The fact that, for many non-Newtonian fluids, the equations of rectilinear motion constitute an overdetermined system for determining the speed was noted by the reviewer [see the paper reviewed below]. A natural conclusion is that, in general, the motion of such fluids through cylindrical tubes will not be rectilinear. These authors point out that the overdetermination can be eliminated by superimposing a plane motion in planes perpendicular to the rectilinear motion. For a particular fluid, they obtain an approximate solution of the resulting equations for flow through an elliptical tube, finding that the superimposed plane motion consists of four similar vortex-like flows. They discuss symmetry considerations which make it seem likely that, for any tube possessing two orthogonal planes of symmetry, the motion is qualitatively similar to this when it is not rectilinear, whatever be the fluid. *J. L. Ericksen* (Washington, D.C.).

Ericksen, J. L. Overdetermination of the speed in rectilinear motion of non-Newtonian fluids. Quart. Appl. Math. **14** (1956), 318–321.

Dans la théorie de Rivlin des milieux continus [Nature **160** (1947), 611; MR **9**, 215] l'Auteur considère le cas des mouvements rectilignes.

Il montre que pour déterminer la vitesse on obtient deux équations aux dérivées partielles, qui ne sont compatibles que si les deux scalaires intervenant dans les équations de Rivlin vérifient une relation, non satisfaite dans le cas général.

L'Auteur termine en discutant brièvement cette condition de compatibilité et, en particulier, il retrouve que parmi les mouvements de l'espèce envisagée figurent ceux déjà signalés par Stone [voir l'article analysé ci-dessous], pour lesquels les lignes d'égale vitesse sont des cercles ou des droites. *R. Gerber* (Toulon).

Stone, D. E. On non-existence of rectilinear motion in plastic solids and non-Newtonian fluids. Quart. Appl. Math. **15** (1957), 257–262.

Rivlin's theory of incompressible fluids is defined by the constitutive equation

$$(*) \qquad t_{ij}=-p\delta_{ij}+F_1 d_{ij}+F_2 d_{ik}d_{kj},$$

where F_1 and F_2 are analytic, or at least continuous, functions of the principal invariants II, III of the rate of deformation tensor d_{ij}. The author adopts (*) but imposes the requirement that the stress be unaffected by uniform changes in speed. This results in restrictions on the form of F_1 and F_2 which exclude most types of fluids, including the classical Navier-Stokes case, but yield forms of F_1 and F_2 that are singular at the state of rest. The resulting theory is similar to that of perfectly plastic solids, but no yield condition is assumed. The author considers steady rectilinear flow in a cylinder of arbitrary cross-section, the material being supposed to adhere to the walls. Following up an analysis of Ericksen [see the paper reviewed above] the author shows that this motion is generally not possible. Specifically, he confirms Ericksen's conjecture that in a non-rigid motion of the type assumed the surfaces of constant speed are planes or portions of right circular cylinders. The author attempts to determine those special fluids, defined by special forms of F_1 and F_2, such that a rectilinear motion is possible in a cylinder of any cross-section. *C. Truesdell.*

Chester, W. The effect of a magnetic field on Stokes flow in a conducting fluid. J. Fluid Mech. **3** (1957), 304–308.

Author's summary: Low Reynolds number flow of a conducting fluid past a sphere is considered. The classical Stokes solution is modified by a magnetic field which, at infinity, is uniform and in the direction of flow of the fluid.

The formula for the drag is found to be

$$D=D_S\left\{1+\frac{3}{8}M+\frac{7}{960}M^2-\frac{43}{7680}M^3+O(M^4)\right\},$$

where D_S is the Stokes drag and M is the Hartmann number. *K. C. Westfold* (Pasadena, Calif.).

Nardini, Renato. Su un gruppo di casi relativi ad onde magneto-idrodinamiche non omogenee. Ann. Mat. Pura Appl. (4) **43** (1957), 371–397.

The author expresses in curvilinear coordinates the magnetohydrodynamic equations of a perfectly conducting, incompressible fluid immersed in a stationary magnetic field. He examines the conditions under which it is possible to uniquely determine the magnotehydrodynamic field, and treats particular cases of cartesian and cylindrical coordinate systems. *C. H. Papas.*

Prakash, Surya; and Tandon, J. N. On the reflection and refraction of magneto-hydrodynamic waves. Proc. Nat. Inst. Sci. India. Part A. **23** (1957), 264–273.

From the authors' summary: – Previous work of Ferraro [Astrophys. J. **119** (1954), 393–406; MR **15**, 761] and Roberts [ibid. **121** (1955), 720–730; MR **16**, 1076] is extended to the case where there are different permanent magnetic fields in the two media. Reflection and refraction are possible only if the discontinuity in the permanent magnetic fields is perpendicular to the plane of incidence and the incident wave is polarized perpendicular to the plane of incidence. The amplitude relations are then the same as Ferraro's, but the laws of reflection and refraction depend on the orientations of the magnetic field in the two media. The interface remains undisturbed by the magneto-hydrodynamic waves. *K. C. Westfold.*

See also: Numerical Methods: Eichelbrenner. **Probality:** Heaps; Vorob'ev. **Mechanics of Particles and Systems:** Pavlenko. **Statistical Thermodynamics and Mechanics:** Margenau. **Elasticity, Plasticity:** Huan Tun. **Optics, Electromagnetic Theory, Circuits:** Sturrock; Jeffreys and Lapwood. **Quantum Mechanics:** Möglich. **Geophysics:** Sekerž-Zen'kovič.

Optics, Electromagnetic Theory, Circuits

Feder, Donald P. Automatic lens design methods. J. Opt. Soc. Amer. **47** (1957), 902–912.

In the course of designing an optical system, one is frequently confronted in principle with the task of reducing to zero a certain number of functions f_k ($k=1, \cdots, M$) of the parameters specifying the constitution of the system. Thus, the f_k might be aberration coefficients and M is then finite only if the effects of aberrations beyond a certain order are known to be negligible. In general the equations $f_k=0$ are incompatible. The author therefore replaces the original task by that of determining the values of the parameters for which a merit function $\phi=\sum \mu_k f_k^2$ (the μ_k are arbitrary positive weighting factors which must somehow be chosen) is a minimum [cf. the following review]. Several methods for treating this problem are discussed. They fall essentially under two main headings: (i) the method of steepest descent, (ii) linearization methods. In the first of these, one starts with an initial 'point' P_0 and computes ϕ_0 and $(\text{grad}\,\phi)_0$. One then proceeds to a neighbouring point P_1, whose coordinates differ from that of P_0 by an amount $-h_0(\text{grad}\,\phi)_0$, ($h_0$ is a small positive number), so that at P_1 one has $\phi_1=\phi_0-h_0|\text{grad}\,\phi|_0^2<\phi_0$. From P_1 one may then proceed to a new point P_2 in the same way, such that $\phi_2<\phi_1$, and so on. The linearization method is a generalization of the Newton-Raphson method for one equation and one variable, so long as the number of equations is equal to the number of unknowns. If the former exceeds the latter, recourse may be had to a least squares method. Finally the method of constraints is discussed, which is intended to take into account the fact that certain of the f_k may be very much more slowly varying functions of the parameters of the system than others. Some numerical examples are provided. The author, in his introduction,

expresses the opinion that it appears "···inevitable that within a very short time all serious design work will be done by machine." This remarkable claim would not seem to be borne out by this paper, not least because of the arbitrariness of the choice of performance functions, weighting functions and the like. As the author remarks finally, "Unfortunately actual computation is capable of providing many surprises, mostly unpleasant···".

H. A. Buchdahl (Hobart).

Feder, Donald P. Calculation of an optical merit function and its derivatives with respect to the system parameters. J. Opt. Soc. Amer. **47** (1957), 913–925.

From the standpoint of geometrical optics the performance of an optical system is specified by certain numbers ("performance numbers"), e.g. aberration coefficients, whose magnitude gives a measure of the quality of the image, the system being perfect when all the numbers are zero. For subsequent use in the context of automatic computing machinery [see preceding review], the author of this paper seeks to define a function H which gives a performance number for any selected ray, and to obtain equations for the derivatives of H with respect to the parameters specifying the constitution of the system. The function chosen here is $H=P-W$, where P is the optical distance between the object point O and the point A' in which a principal ray intersects a chosen image plane, whilst W is the optical distance between O and the point B', which is the foot of the normal from A' on to some arbitrary ray. H is calculated from ray traces, and the ray tracing equations are given in detail in a form which is appropriate to the calculation of the derivatives of H, which are considered later in the paper. The results relating to differential changes are claimed to be exact in the case of meridional rays, but only approximate for skew rays. Under disregard of chromatic aberrations, distortion, boundary conditions, glass restrictions and the condition relating to focal length, a merit function ϕ_H may be defined as

$$\phi_H=\Sigma(H/w_H s_0^2)^2,$$

where w_H is a weighting factor and s_0 the semiaperture for the ray in question, the summation being extended over all the rays for which H has been computed. When the various features disregarded above are taken into account, a merit function ϕ is defined for each of them, and the total merit function $\Phi=\Sigma\phi$. The kind of approximation involved in the calculation of dH for skew rays is discussed in terms of a numerical example.

H. A. Buchdahl (Hobart).

Bergstein, Leonard; and Motz, Lloyd. Third-order aberration theory for varifocal systems. J. Opt. Soc. Amer. **47** (1957), 579–583.

The effect of bending one of the component lenses on the Seidel sums of a lens system is investigated. The formulas are applied to a four component varifocal system. *E. Wolf* (Manchester).

Gontier, Gérard. Sur la précision des relations utilisées pour l'application de l'interférométrie différentielle aux mesures de l'épaisseur optique d'une lame transparente isotrope. C. R. Acad. Sci. Paris **244** (1957), 2591–2594.

Hopkins, H. H. Applications of coherence theory in microscopy and interferometry. J. Opt. Soc. Amer. **47** (1957), 508–526.

The paper presents a review of the theory of partial coherence in optics and discusses some of its more important applications. These include the theory of the stellar interferometer and the interference microscope, the effect of the source size on the visibility of fringes in two-beam interferometers, and the influence of the numerical aperture of the illuminating system on resolution of a microscope.

Whilst the validity of results which relate to practical applications is not doubted, the discussion of the basic theory is, in the reviewer's opinion, not quite satisfactory. An impression is given that partial coherence arises entirely from phase fluctuations, and amplitudes are treated as constants. This approach seems to lead to the formula (quoted on p. 508) $\langle\cos(\delta_2-\delta_1)\rangle=V_{21}\cos(\delta_2-\delta_1+\beta_{21})$ where δ_1 and δ_2 are supposed to be random functions of time, V_{21} and β_{21} are constants and sharp brackets denote time average. The meaning of this relation is not clear to the reviewer, since the expression on the left is time independent, whilst the expression on the right seems in general to depend on time. Further, the use of the term "monochromatic" is rather misleading. If it is understood in the sense as generally accepted by mathematical physicists (infinitely long trains of a sharp frequency), a statement such as "monochromatic components of disturbances arising at different atoms are incoherent" (p. 511) is not correct. The demonstration of this result, given in section 2 is in the reviewer's opinion invalid, since it does not seem possible to represent a strictly monochromatic strain as a finite series of pulses, emitted at randomly distributed time instants (as demanded by formula (8)). Readers interested in the rather difficult questions concerning the basic theory of partial coherence should consult a paper by A. Blanc-Lapierre and P. Dumontet [Rev. Opt. **34** (1955), 1–21; MR **16**, 884].

E. Wolf (Manchester).

Dumontet, Pierre. Calcul de l'amplitude d'une onde plane réfléchie par une lame métallique homogène à faces parallèles. C. R. Acad. Sci. Paris **244** (1957), 2234–2236.

The reflection and transmission coefficients for normal incidence of a plane parallel conducting film is calculated on the basis of the electron theory of metals. Electron collisions are taken into account. It appears that in the visible range, the coefficients for films of silver, aluminum and gold are, irrespective of their thickness, in good agreement with the predictions of the classical theory of Sommerfeld. *E. Wolf* (Manchester).

Jánossy, L. On the classical fluctuation of a beam of light. Nuovo Cimento (10) **6** (1957), 111–124.

The fluctuations in the intensity of a quasi-monochromatic light beam from a macroscopically constant source are investigated. The source is regarded as consisting of a number of damped oscillators, and the generating function of the basic distribution is calculated. It is found that its moments are, to a good approximation, the same as that of a Gaussian distribution. The results have a bearing on some recent experiments on photon coincidences, which are briefly discussed. *E. Wolf.*

★ **Blanc-Lapierre, A. Sur quelques applications de la théorie des fonctions aleatoires à l'optique.** Proceedings of the International Congress of Mathematicians, 1954, Amsterdam, vol. III, pp. 339–347. Erven P. Noordhoff N.V., Groningen; North-Holland Publishing Co., Amsterdam, 1956. $7.00.

An exposition of some known results concerning

applications of the theory of stochastic processes to optics. These include some remarks on classical radiation theory, some results on partial coherance of luminous sources and on polarization. For example, using the Poincaré spherical representation of polarization states, there is the ergodic theorem: Let D be a domain on the Poincaré sphere, $P(m \in D)$ the probability that the point representing state m is in D, $2\Delta T$ the measure of the set of time between $-T$ and T such that $m \in D$, then $\Delta T/T \to P(m \in D)$ as $T \to \infty$. *G. L. Walker* (Southbridge, Mass.).

Slevogt, H. Zur Beugungstheorie der Wellenflächen mit nur einer Symmetrie-Ebene. Optik **14** (1957), 377–387.

Wave fronts possessing a single plane of symmetry are considered and an expansion of the complex wave function is developed in terms of the circle polynomials of Zernike [Physica **1** (1934), 689–704]. This generalizes a familiar expansion in terms of Legendre polynomials for rotationally symmetric wave fronts. The result is applied to Zernike's Strehl Definition for optical image evaluation and to wave front aberrations. Of course, even in the third order theory, the classification of types of aberrations becomes more complicated than in the rotationally symmetric case. *G. L. Walker* (Southbridge, Mass.).

Savelli, Michel. Etude d'un modèle suggéré par des recherches sur les propriétés statistiques de la transparence des émulsions photographiques. C. R. Acad. Sci. Paris **244** (1957), 1710–1712.

Jancel, Raymond; et Kahan, Théo. Conditions de décharge dans une cavité électromagnétique et ondes progressives dans les plasmas lorentziens. C. R. Acad. Sci. Paris **244** (1957), 2894–2896.

Toraldo di Francia, G. On a macroscopic measurement of the spin of electromagnetic radiation. Nuovo Cimento (10) **6** (1957), 150–167.

The author undertakes calculation of the angular momentum absorbed by an infinitesimally thin circular screen exposed to an incident electromagnetic wave with elliptic polarization. The screen is assumed to possess infinite and null conductivities parallel and perpendicular to a given direction in its plane. An integro-differential equation for the unidirectional current in the screen is solved by series expansion in reciprocal powers of the wave length, assumed large compared to the screen radius. A complex scattering cross section is defined and a few terms of its expansion at long wave lengths are given, where the real part (ordinary scattering cross section) is much smaller than the imaginary part. The latter circumstance makes for complication in the accurate measurement of the spin of a circularly polarized wave, because the effect of slight ellipticity of polarization is marked. *H. Levine* (Stanford, Calif.).

Williams, W. E. Reflection and refraction of electromagnetic waves at plane interfaces. J. Math. Phys. **36** (1957), 26–35.

In this note, the author presents a unified method of solving the problem of the reflection and transmission of an arbitrary monochromatic electromagnetic disturbance by a dielectric slab. There are three dielectric media to be considered; the region 1, $z>0$, in which the disturbance originates, the region 2, $-a<z<0$, which is the dielectric slab, and the region 3, $z<-a$, into which the disturbance is transmitted. If the region j has conductivity σ_j, dielectric constant ε_j, permeability μ_0, each component of the electric and magnetic vectors in that region satisfies the equation $(\nabla^2+k_j^2)\Psi=0$ where $k_j^2=\omega\mu_0(\omega\varepsilon_j\varepsilon_0-i\sigma_j)$, the time factor being $e^{i\omega t}$ and the dielectric constant in vacuo being ε_0.

The method used is an extension of the integro-differential equation technique which has been developed for discussing diffraction of plane electromagnetic waves by an aperture in a perfectly conducting plane screen. *E. T. Copson* (St. Andrews).

Moriguchi, Nobuo. Theory of non-resonant slotted antenna in sidewall of rectangular waveguide. Bull. Univ. Osaka Prefecture, Ser. A. **4** (1956), 67–79.

The general theory is developed for nonresonant slots in the broad or narrow face of a wave guide. Formulas are given for both real and imaginary parts of the conductance or the impedance. In particular computations are given for the admittance of a centered transverse slot as a function of the slot dimensions, and of the impedance of a longitudinal slot also as a function of slot dimensions. The latter computation assumes an offset equal to $a/4$ where a is the width of the guide. The computations and theory assume a negligible wall thickness.

The method breaks the region of interest into three regions, with imaginary surfaces in common between each pair of regions. The modes in the wave guide are expressed in the usual manner but in a region just under the slot transverse wave guide modes are utilized. Matching on the common boundaries yields an integral equation. The author's technique avoids much of the convergence problem present in the more usual Schwinger integral equation formulation of the problem and leads to an integral equation which can evidently be solved without variational technique. *W. K. Saunders.*

Beckmann, Peter; und Franz, Walter. Berechnung der Streuquerschnitte von Kugel und Zylinder unter Anwendung einer modifizierten Watson-Transformation. Z. Naturf. **12a** (1957), 533–537.

The authors have computed the cross section of large conducting spheres and cylinders through the use of the Watson transformation applied to the traditional eigenfunction series. They find that with both sphere and cylinder, the cross section may be developed in an asymptotic series in powers of $(ka)^{2/3}$, where k is the wave number and a the radius. The technique appears relatively straightforward and one is surprised to find that this path has not been successfully pursued sometime during the past forty years. *W. K. Saunders.*

Agdur, Bertil. On the interaction between microwave fields and electrons, with special reference to the strophotron. Ericsson Technics **13** (1957), 3–108.

The author examines a non-linear problem in which electrons interact with a micro-wave field in a strophotron. This is a device which gives a new solution to the question of how to vary the frequency of a microwave oscillator by electronic means. The author also describes the experimental work on the device. *V. M. Papadopoulos.*

Mawardi, Osman K. On the concept of coenergy. J. Franklin Inst. **264** (1957), 313–332.

The author extends the well-known analogy between mechanical and electrical quantities whereby the dynamic equations of an electrical network may be written down with ease from the Lagrangian of the system. The La-

grangian method is also valid for electromechanical systems when no magnetic or dielectric materials are present in the electromagnetic field. The author introduces a work function (co-energy) related to the magnetic and electric energies in the presence of magnetic or dielectric materials, and he shows that the methods of analytic dynamics are generally applicable. He also gives the statistical mechanics approach to the same problem.

V. M. Papadopoulos (Providence, R.I.).

Hofmann, H. Über den Kraftangriff des allgemeinen, nichtstationären elektromagnetischen Feldes an der Materie. Österreich. Ing.-Arch. **11** (1957), 123–134.

Continuation of MR **18**, 536, 847.

Sturrock, P. A. Non-linear effects in electron plasmas. Proc. Roy. Soc. London. Ser. A. **242** (1957), 277–299.

The fluctuations in an electron plasma are governed by non-linear equations, even in the simple model studied (unboundedness, uniform ion density, zero temperature, zero external fields, immobility of the ions, no collisions, non-relativistic velocities, no multiple streaming). Corrections to the linear approximation are developed in powers of a "non-linearity" parameter, successive coefficients being of successively higher order in the "depth of modulation" D, defined as the mean electron density fluctuation relative to the ion density.

The fluctuations are Fourier analyzed in space and the non-linear corrections to the linear approximation of the Fourier components (harmonic time variation at plasma frequency) are given as secular effects on their amplitudes and phases. Mutual coupling of Fourier components (uncoupled in the linear approximation) is obtained.

If the disturbances are one-dimensional, no effect on the amplitudes is found, to the order D^3 (inclusive): there is no coupling between different spatial Fourier components as regards energy exchange. Sturrock calls this "coherence". Only the phases are related to the wave-numbers: there is a dispersion law.

Two- and three-dimensional disturbances exhibit amplitude effects of the order D^2: energetic coupling between four different wave numbers can occur (when these are not equal in pairs). This is called "incoherent interaction". The case of almost unidirectional two-dimensional spectra is analyzed in detail. It is found that these spectra spread transversely into a uniform band of increasing width, an initial small angular spread ε becoming appreciable within $(4\varepsilon D^2)^{-1}$ plasma periods, showing a tendency towards isotropy.

While the language of Fourier analysis and spatial spectra illustrates the results in familiar terms, Fourier analysis is not necessarily the most suitable method in a non-linear problem, and in the calculation just mentioned, Sturrock has to reconstitute a Fourier series into an auxiliary function which to the reviewer appears to be the velocity potential, in non-analyzed form.

By avoiding Fourier analysis it is possible also to answer Sturrock's questions whether in one-dimensional disturbances shock-fronts arise and at what order beyond D^3 spectral decay takes place. {The reviewer found no effect to all orders, and no shock fronts, in that strictly harmonic oscillation of all electron layers, with phase and amplitude distributions which do not lead to multiple streaming, represent exact solutions of the one dimensional non-linear equations.}

A further analysis of the interaction between longitudinal plasma oscillations and transverse unmodulated motions is carried out and followed by an extensive discussion of the results in terms of various physical concepts, such as harmonics, traveling waves, Landau damping, viscosity, plasmon collisions. The theory of transverse spectral broadening is found to agree with observations.

O. Buneman (Stanford, Calif.).

Bunkin, F. V. On radiation in anisotropic media. Soviet Physics. JETP **5** (1957), 277–283.

A general solution is obtained for the field due to a prescribed monochromatic current distribution in an infinite homogeneous medium of arbitrary anisotropy. The method consists of finding a Green's tensor (fundamental singularity), which when multiplied by the current density and integrated over the volume occupied by the source distribution gives the field. For the radiation field, it is sufficient to obtain an asymptotic approximation to this integral, valid for the wave zone. The method of steepest descents is used to calculate the Green's tensor in the case of a magnetoactive (e.g., magneto-ionic) medium and the multipole expansion of the radiation field is derived. In particular, the field due to an electric dipole is considered in some detail.

K. C. Westfold (Pasadena, Calif.).

Babič, V. M. Ray method for calculation of intensity of wave fronts. Dokl. Akad. Nauk SSSR (N.S.) **110** (1956), 355–357. (Russian)

The author, basing himself on previous considerations [Vestnik Leningrad. Univ. **11** (1956), no. 1, 107–124; MR **19**, 197], describes a method for calculating the intensity of wave front of longitudinal and transverse waves in a non-homogeneous medium and for the intensity of a wave front reflected from curved boundaries. Essentially, the method depends on the fact that, in the small, a non-homogeneous medium may be taken to be homogeneous, and a curved wave front may be taken to be plane. Let the scalar or vector function $u(x, y, z, t)$ represent the wave phenomenon. It is assumed initially that

$$u=u_0(x, y, z)f_0(t-\tau)+u_1(x, y, z)f_1(t-\tau)+O(f_2(t-\tau)),$$

where $f_2'=f_1$, $f_1'=f_0$, $|O(f_2)|<c|f_2|$, $|f_1|\gg|f_2|$, $|f_0|\gg|f_1|$, $c=$const; the wave front intensity being $|u_0|$. Results of S. L. Sobolev [Trudy Seismol. Inst. no. **6** (1930)] for the wave equation; and of K. O. Friedrichs and J. B. Keller [J. Appl. Phys. **26** (1955), 961–966; MR **17**, 553] for a concrete problem in a non-homogeneous medium, are obtained as special instances of the procedure outlined here.

J. B. Diaz (College Park, Md.).

Jeffreys, Harold; and Lapwood, E. R. The reflexion of a pulse within a sphere. Proc. Roy. Soc. London. Ser. A. **241** (1957), 455–479.

A pulse sets out from an interior point of a uniform compressible liquid sphere. The formal solution for the velocity at a surface point at any subsequent time is derived by operational methods. This solution, having the form of an infinite series of Legendre polynomials, with coefficients which are the ratios of Bessel functions, converges too slowly to be useful.

A modification of a method of van der Pol and Bremmer is used, in which the first part of the series is transformed by Watson's device into a series of multiple contour integrals, which are evaluated at saddlepoints. The expressions thus derived are shown to correspond term by term to the pulses whose times of arrival can be calculated

by geometrical ray theory. The shape of each arriving pulse, not obtainable by ray theory, is given by the present method.

It is shown that stationary time pulses which are extremals are related to stationary time pulses which are not extremals as Fourier integrals to the allied Fourier integrals. In particular, it is shown that the two singly-reflected pulses which travel within the shorter arc of a great circle are so related. If the initial disturbance is given by a Heaviside unit function, the minimum time pulse arrives as a unit function, but the stationary time pulse which is not extremal arrives gradually and rises to a logarithmic infinity. *A. E. Heins* (Pittsburgh, Pa.).

Poincelot, Paul. Sur la réflexion ionosphérique en présence de chocs. C. R. Acad. Sci. Paris **244** (1957), 2298–2299.

Friis, H. T.; Crawford, A. B.; and Hogg, D. C. A reflection theory for propagation beyond the horizon. Bell System Tech. J. **26** (1957), 627–644.

Karp, S. N.: and Williams, W. Elwyn. Equivalence relations in diffraction theory. Proc. Cambridge Philos. Soc. **53** (1957), 683–690.

A prepublication report of this paper [Div. Electromag. Res., Inst. Math. Sci., New York Univ., Res. Rep. No. **EM-83** (1955), i+16 pp.] is reviewed in MR **18**, 170.

A. E. Heins (Pittsburgh, Pa.).

Karp, S. N.; and Radlow, J. On resonance in infinite gratings of cylinders. Div. Electromag. Res., Inst. Math. Sci., New York Univ., Res. Rep. No. **EM-90** (1956), i+40 pp.

In this paper the authors have investigated the behavior of diffraction gratings of cylinders near certain critical frequencies which produce resonance effects. The treatment is based on previous studies [same Rep. No. **EM-75** (1955); **EM-85** (1955); MR **17**, 433, 1029; Karp, Scattering by a combination of obstacles, McGill Univ. Symposium, June, 1953], where the non-resonance case has been worked out for arbitrarily shaped obstacles and unrestricted wavelengths. A summary of these results is presented in section 2 of the report, since they are essential to the present analysis. The total field, being regular in the exterior of a horizontal strip containing the grating elements, is expressed by the formula

$$\text{(i)}\quad u=u_{inc}+\sum_{-\infty}^{\infty}\frac{2}{kd\sin\theta_\nu}f(\pm\theta_\nu)\exp[ikr\cos(\theta\mp\theta_\nu)],$$

where θ is the observation direction and θ_ν are the spectral angles depending on k, d (spacings) and the angle of incidence, φ. The function $f(\theta)$ is the angular distribution function. The problem then is to determine $f(\theta)$ for values of $\theta=\theta_\nu$ and real, which are the critical directions resulting in resonance. The angular distribution function $f(\theta)$ is expressed in the form

$$\text{(ii)}\qquad f(\theta)=f_I(\theta)+A_Lf(0)f_L(\theta)+A_Rf(\pi)f_R(\theta),$$

which describes the far fields of a single scatterer in the grating when excited by the incident field and the two incoming waves from $x=-\infty$ and $x=\infty$, respectively. The coefficients A_L and A_R are known expressions which depend on the angle of incidence, the grating spacing, and the frequency. For the two values of $\theta=(0,\pi)$ the denominator of the second term of (i) vanishes, and therefore it is necessary to investigate the properties of the angular distribution $f(\theta)$ for these critical values. From (ii) $f(0)$ and $f(\pi)$ are expressed in terms of the known values of f_I, f_L and f_R evaluated at 0 and π and the constants A_L and A_R. They are expressed in terms of certain determinants α_1, α_2, α_3 involving the known f_I, f_L and f_R. For the critical values of $\theta=(0,\pi)$ either one of the quantities A_L, A_R or both diverge. This situation arises when either (a) $kd(1-\cos\varphi)$, or (b) $kd(1+\cos\varphi)$ approaches an integral multiple of 2π, and case (c) when both cases (a) and (b) approach integral multiples of 2π simultanously.

In section 3 the authors examine the behavior of the constants α_1, α_2, α_3. It is shown that if α_3 vanishes then α_1 and α_2 also vanish. The necessary and sufficient condition for $\alpha_3=0$, for the case $E_{||}$, is that the boundary be composed of one or more zero-thickness vertical strips (perfect conductors) spaced at half integral multiples of the wavelength. For $H_{||}$ one could add any number of horizontal strips, i.e., a rectangle of suitable width. For case (c) and for $\alpha_3\neq0$, it is shown that $f(\pm\theta_\nu)/kd\sin\theta_\nu$ approaches a finite value as the resonance case is approached, and if $\alpha_3=0$ this ratio also approaches a finite value up to order $(kd)^{-1}$. Then follows a brief discussion of energy conservation. In the last section the previous results are compared with those obtained by Artmann [Z. Physik **119** (1942), 529–567], whose analysis is based on small corrugations compared to wavelength and arbitrary spacings. To make this comparison the authors have specialized their results to small thickness; then with the restriction that kd is large but no restriction on the thickness when compared to the wavelength, the results agree with those of Artmann, where the thickness is small in comparison to the wavelength but large spacings. The calculations of the results given in the text are to be found in the appendices. *N. Chako.*

Kay, I.; and Moses, H. E. Reflectionless transmission through dielectrics and scattering potentials. Div. Electromag. Res., Inst. Math. Sci., New York Univ., Res. Rep. No. **EM-91** (1956), i+16 pp.

The problem of constructing a plane stratified dielectric medium through which a plane polarized monochromatic wave can pass without reflection is analyzed. This problem is not unlike the one discussed by V. Bargmann [Rev. Mod. Phys. **21** (1949), 488–493; MR **11**, 248] in connection with the radial Schrödinger equation with zero angular momentum. *C. H. Papas* (Pasadena, Calif.).

Takahashi, Isao; Watanabe, Tadashi; and Tanimoto, Kiichi. Diffraction of electromagnetic waves by a circular cylinder of infinite length. Mem. Coll. Sci. Univ. Kyoto. Ser. A. **28** (1957), 209–219.

With the aid of a variational principle, the authors calculate the scattering cross-section σ of a perfectly conducting circular cylinder of radius a. The incident wave is TEM and polarized perpendicular to the axis of the cylinder. At low frequencies the expression they obtain for σ by a variational calculation checks with the exact expression one would obtain by the classical Fourier-Lamé method. In the limit of high frequencies they find that $\sigma=4a$. *C. H. Papas* (Pasadena, Calif.).

Landauer, R. Spatial variation of currents and fields due to localized scatterers in metallic conduction. IBM J. Res. Develop. **1** (1957), 223–231.

Localized scatterers can be expected to give rise to spatial variations in the electric field and in the current distribution. The transport equation allowing for spatial

varations is solved by first considering the homogeneous transport equation which omits electric fields. The homogeneous solution gives the purely diffusive motion of current carriers and involves large space charges. The electric field is then found, and approximate space charge neutrality is restored, by adding a particular solution of the transport equation in which the electric field is associated only with space charge but not with a current. The presence of point scatterers leads to a dipole field about each scatterer. The spatial average of a number of these dipole fields is the same as that obtained by the usual approach which does not explicitly consider the spatial varation. Infinite plane obstacles with a reflection coefficient r are also considered. These produce a resistance proportional to $r/(1-r)$. (Author's abstract.)

P.-O. Löwdin (Uppsala).

Phariseau, P. The diffraction of light by two perpendicular ultrasonic waves. II. Physica **23** (1957), 1103–1114.

Referring to our previous paper [MR **19**, 608], we deduce solutions in the form of a series of Bessel functions for the systems of difference-differential equations, appearing there. After giving rigorous relations between the coefficients we are able to give explicit expressions for the intensities of the different spectra in the case of small widths of the ultrasonic field. Investigating the extremum of these intensities for variable angles of incidence, we find that they become maximum when the angle of incidence is equal to the Bragg angle. So we have proved the complete analogy between the diffraction of light by two perpendicular supersonic waves and the X-ray diffraction in crystals. *Author's summary.*

Lin, Wei-Guan. Contributions to the theory of non-ideal waveguides. Sci. Record (N.S.) **1** (1957), no. 2, 27–32.

★ **Lewis, Philip M., II. The synthesis of voltage transfer functions.** Research Laboratory of Electronics, Massachusetts Institute of Technology, Cambridge, Mass., Tech. Rep. 314 (1956). iii+105 pp.

Voltage transfer functions in the realm of finite passive reciprocal networks without mutuals have been synthesized previously by A. Fialkow and I. Gerst [J. Math. Phys. **30** (1951), 49–72; Proc. I.R.E. **43** (1955), 462–469; Quart. Appl. Math. **12** (1954), 117–131; MR **13**, 190; **16**, 1182]. This report, while paralleling this and related work, systematizes and generalizes the subject and evolves various simplified synthesis procedures. As the general form of the transfer function $A(s)$ (s—complex frequency variable) is known: $A=KD/N$, where $N(s)$ — Hurwitz polynomial with real coefficients of degree n, $D(s)$ — real polynomial of degree $\leq n$, both with leading coefficient 1 and K — real transfer constant, realizability theorems concern solely K_{max} and restrictions on $D(s)$. A complete and nonredundant set: Absence of roots of $(D\pm KN)$, and of $D(s)$ in case of grounded networks, on the positive real axis; non-negativity of the coefficients of both non-contracted transfer functions of grounded, and of $(D\pm KN)$ of general networks; plus a corollary on the minimum phase of roots in terms of degree, implying minimal augmentation, is cogently chosen; proofs are elementary. Definitions of, and tests for, critical K evolve. Lattice realization of increasing generality, including numerous special treatments and short-cuts, is treated in detail, followed by lattice-unbalancing procedures. Beyond the realm of lattice realizability, general synthesis procedures by means of multiple-tree-type networks are presented, along the lines of Fialkow and Gerst, but offering some generalizations and simplifications. The paper contains numerous errors, none basic, but hampering its perusal.

H. G. Baerwald (Cleveland, Ohio).

★ **Proceedings of the symposium on nonlinear circuit analysis, New York, April 25, 26, 27, 1956. Vol. VI.** Polytechnic Institute of Brooklyn, Brooklyn, N.Y., 1957. xiii+456 pp.

The first paper, "Stability Theory" by W. Kaplan, is a survey of recent work on stability theory of ordinary differential equations with an extensive bibliography. In "Engineering Applications of Non-Linear Theory" by N. W. McLachlin, a summary of methods of solution and engineering applications of specific non-linear ordinary equations together with references is given. The relation between initial conditions and the subsequent harmonic and subharmonic oscillations is studied in "Initial Conditions for Certain Types of Non-Linear Oscillations" by Chihiro Haryashi using phase plane techniques. In "Three Frequency Resonance" by E. Pinney, a vacuum tube circuit problem is formulated as a non-linear system of ordinary differential equations, and the asymptotic behaviour at resonant frequencies is treated by methods based on the general theory of systems of non-linear differential equations.

The describing function method is used for non-linear elements whose output is a prescribed function of the input. If the input-output relation of a non-linear element involves derivatives this technique is not applicable. A generalization for defining the first harmonic, which applies to the general case and is based on Hamilton's integral, is given by K. Klotter in "An Extension of the Conventional concept of the Describing Function". An interesting and novel approach to the optimazation problem for non-linear control systems which uses the Theory of Dynamic Programming is presented by R. Bellman in "On the Application of the Theory of Dynamic Programming to the Study of Control Processes". These techniques are applied to stochastic control processes and to "bang-bang" control.

The extension of phase-plane methods to higher dimensional phase spaces is considered in "A Mathematical Study of On-Off Controlled Higher-Order Systems" by B. Hamel. Results concerning the existence of a limit cycle are obtained for a system which is linear except for one term involving on-off control.

In "Optimum Design of Final-Value Control Systems" by R. C. Booton Jr., the optimazation of the root-mean-square position error of a control system subject to a root-mean-square velocity error constraint is studied. The linear case is considered first in discrete form. The results for the linear system are used as a basis for treating a quasi-linear system.

The paper by R. Deutsch "On the Distribution of Nonlinear Circuit Transformation" discusses the use of integral and partial differential equations in the solution of non-linear circuit problems for which the inputs are defined by their statistical properties, and some examples are given.

A procedure for relating the open and closed loop characteristics of non-linear feedback systems which contain a computer in the loop is given by J. R. Burnett in "Digital Computers in the Synthesis of Nonlinear Feedback Systems". This method is based on the analytic representation and approximation by means of poly-

nomials of a multi-valued sequence generator. In "Nonlinear Aspects of Sampled Data Control Systems" by R. E. Kalman, a phase-plane analysis of a class of sampled-data control systems is carried out. A formulation in terms of stochastic processes is given and an equivalence with continuous systems is established for a subclass.

An algebra to facilitate the study of networks containing elements with piecewise linear characteristics is presented in "Piecewise-Linear Network Analysis and Synthesis" by T. E. Stern, and applications to the synthesis of driving point impedances for networks containing such elements as diodes as well as to function generators are given. In "A Note on the Analysis of Flip-Flop" by L. M. Vallese, a phase-plane analysis of a canonic flip-flop is given, where a canonic flip-flop is defined as a system possessing at least two stable singularities separated by a saddle point. The paper "The Nonlinear Resistive 3-Pole: Some General Concepts" by W. Millar concerns a generalization of the concepts of reciprocity and anti-reciprocity from linear to non-linear networks based upon group-theoretic considerations. Applications to synthesis of gyrators and K elements, and the study of the effects of groups of linear transformations of terminals and Hamiltonian transformations are given. E. Weber presents the application of Laplace transform to the non-linear ordinary differential equation which contains powers of the unknown function in "Complex Convolution Applied to Nonlinear Problems". Transforms of these powers are computed by the use of the complex convolution. The determination of the frequency spectrum and amplitudes is based on the partial fraction expansion of the unknown function which is assumed to be meromorphic.

An application of almost periodic functions is used to analyse the power dissipation in a non-linear resistor for applied emf's which are Stepanoff-almost-periodic and the results are applied to modulation in "Frequency Conversion with Positive Nonlinear Resistors" by C. H. Page. The remaining articles which are not mentioned above, apart from a paper by R. Bass [Amer. J. Math. **77** (1955), 734–742; MR **17**, 611], are chiefly of engineering interest.

C. Saltzer (Cleveland, Ohio).

Lupanov, O. B. On rectifier and contact rectifier circuits. Dokl. Akad. Nauk SSSR (N.S.) **111** (1956), 1171–1174. (Russian)

A (p, q) rectifier circuit is constructed using unilaterial conducting elements with p input nodes and q output nodes such that the admittance between any pair of inputs, any pair of outputs, and from any output to any input is zero. The circuit operation is defined by a p by q admittance matrix. Asymptotic estimates and bounds are determined for the number of rectifiers required to realize such circuits. An application is given to circuits containing rectifiers and contacts which realize logical operations.

C. Saltzer (Cleveland, Ohio).

★ **Lupanov, O. B. On rectifying and rectifying-switching circuits.** Translated by Morris D. Friedman, 572 California St., Newtonville 60, Mass., 1956. 6 pp.

Translation of the paper reviewed above.

Teng, Lee C. Linear theory of betatron oscillations in sectorial cyclotrons. Rev. Sci. Instrum. **27** (1956), 1051–1058.

A transformation to orbit coordinates is used to analyze the linear oscillations in a fixed field alternating gradient accelerator. The normal and axial oscillation frequencies are obtained as a power series in $1/N^2$, where N is the number of sectors in the accelerator; the coefficients in this power series are given explicitly in the appendix up to the third order, $1/N^6$. These coefficients contain three distinct types of terms corresponding respectively to the principle focusing actions present in the Thomas cyclotron, the alternating gradient synchrotron, and the spirally ridged FFAG accelerator. A derivation is provided demonstrating the possibility of designing a magnetic field having fixed oscillation frequencies and a momentum compaction varying over a wide range. An expression is also obtained for the magnetic field strength in the median plane.

D. Schiff (Waltham, Mass.).

★ **Goedecke, Werner. Mathematische Grundlagen der Wechselstromlehre. Band I: Anwendungen des symbolischen Rechenverfahrens auf periodische, rein sinusförmige Wechselvorgänge. Zweite Auflage.** VEB Verlag Technik, Berlin, 1956. 120 pp.

Herzenberg, A.; and Lowes, F. J. Electromagnetic induction in rotating conductors. Philos. Trans. Roy. Soc. London. Ser. A. **249** (1957), 507–584.

A material of uniform conductivity σ filling all space is considered. Part of the material is in rotational motion and the remainder is stationary. A uniform magnetic field is applied (a) parallel and (b) perpendicular to the axis of rotation. The induced magnetic field and the accompanying current systems are calculated. The following examples are treated. For the case (a), semi-infinite cylinder, finite cylinder and sphere rotating with constant angular velocity. The case (in general) of a time-dependent angular velocity is likewise treated, especially the cases of sinusoidal, short pulse, step function and rectangular pulse character of the angular velocity. Examples for the case (b) are the rotating sphere and the rotating finite cylinder at low (constant) angular velocity. Further more treated is the case of a rotating conductor completely embedded in a stationary material filling only the half space and the case of the isolated rotator. Numerical and experimental results are given.

F. Oberhettinger (Madison, Wisc.).

See also: Numerical Methods: Sabroff and Higgins. **Statistical Thermodynamics and Mechanics:** Margenau. **Fluid Mechanics, Acoustics:** Nardini. **Astronomy:** Arsac.

Classical Thermodynamics, Heat Transfer

Matsumae, Shigeo. Criticism on the Fujita's opinions on the thermodynamics. Kumamoto J. Sci. Ser. A. **2** (1956), 351–355.

Fujita, Shigeichi. Quasi-stationary process. III. Changes of state of gas by the slow changes of its boundary conditions. Kumamoto J. Sci. Ser. A. **2** (1955), 277–288.

[For parts I–II see same J. **2**, 51–63 (1954), 174–183 (1955); MR **17**, 567, 1262.] An ideal gas with temperature T is in contact with a heat bath with temperature T_1. The transfer of heat is assumed proportional to T_1-T. This assumption, together with $pv=RT$, determines the state of the gas at all times, when T_1 and either p or v are given as functions of time. Some simple cases are calculated explicitly.

N. G. van Kampen (New York, N.Y.)

Sutton, George W. On one-dimensional heat conduction with an arbitrary heating rate. J. Aero. Sci. **24** (1957), 854–855.

The solution of the elementary boundary value problem $T_t=T_{xx}$, $T(x, 0)=T_0$, $T_x(0, t)=0$, $T_x(1, t)=Q(t)$, is written. Coefficients in a series representing $T(x, t)$ are tabulated for the case in which $Q(t)$ is a polynomial.

R. V. Churchill (Ann Arbor, Mich.).

Selig, F.; und Fieber, H. Bemerkung zu: Wärmeleitprobleme mit zeitlich variabler Übergangszahl. Österreich. Ing.-Arch. **11** (1957), 225–227.

The original paper [Österreich. Ing.-Arch. **11** (1957), 37–40; MR **19**, 213] contained the implicit assumption that the Lamé coefficients g_i for the curvilinear coordinates are homogeneous functions of the second degree. The authors show how this condition can be relaxed.

R. V. Churchill (Ann Arbor, Mich.).

Hofmann, Rudolf. Die Lösung eines speziellen Wärmeleitungsproblems der Elektrotechnik mittels der zweidimensionalen Laplace-Transformation. Arch. Elek. Übertr. **11** (1957), 278–282.

Die rechnerische Behandlung der Erwärmung eines ∞ langen Zylinders mit oder ohne Schutzring durch eine heisse Scheibe führt auf die partielle Differentialgleichung $\partial^2\theta/\partial x^2-\partial\theta/\partial y=0$. Der Verf. zeigt, wie diese mit Hilfe des zweidimensionalen Laplacetransformation bei geeigneten, in der Technik vorkommenden Anfangs- und Randbedingungen leicht gelöst werden kann. *W. Saxer.*

Jain, V. K. On considerations of bore resistance and heat transfer on the internal ballistics of a gun using composite charges. Proc. Nat. Inst. Sci. India. Part A. **23** (1957), 130–142.

Traditionally, bore resistance to travel (in Interior Ballistics) has usually been handled by an assumption equivalent to regarding it as proportional to pressure. Similarly, heat transfer to barrel has usually been accounted for by modifying the value of γ for the propellant. Neither of these simplifying assumptions corresponds closely to the best current evidence concerning the physical phenomena. The author adopts instead the reasonably plausible assumptions that bore resistance, after engraving, is constant, and that heat transfer is proportional to travel. Formulas are computed. No numerical tables are given. *A. A. Bennett.*

Jain, V. K. On internal ballistics of a tapered bore gun using composite charges. Proc. Nat. Inst. Sci. India. Part A. **23** (1957), 108–118.

The author proposes a method of step-by-step numerical integration of basic equations of Interior Ballistics for a conically tapered bore, using composite charges, taking into account covolume heat transfer and bore resistance. A numerical example is presented. *A. A. Bennett.*

Kapur, J. N. The general theory of moderated charges. Proc. Nat. Inst. Sci. India. Part A. **23** (1957), 73–92.

A "moderated charge" (in Interior Ballistics) consists of nominally identical grains, each of which consists of a number of layers of different burning rates. With suitable design, the theory indicates the availability of high piezometric efficiency. Using also composite charges (having grains of different sorts), still further advantages should be secured. The author develops the appropriate formulas. *A. A. Bennett* (Providence, R.I.).

Kapur, J. N. Solution of the equations of internal ballistics when the rate of burning is a linear function of the pressure. Trans. Nat. Inst. Sci. India **3** (1956), 257–289.

The author cites evidence and authority in favor of the (nonhomogeneous) linear law of burning of propellant within gun bores and rockets as against the two more traditional laws. He seeks to give as complete a theory as possible for this more intractable case. Four different procedures are developed, choosing different variables as independent. A specially detailed study is made for the isothermal model with constant burning surface and neglecting nonlinear terms in a dimensionless argument concerned with relative pressure. Special discussion is made of the influence of the nonhomogeneous term of the burning-function on muzzle velocity and maximum pressure. *A. A. Bennett* (Providence, R.I.).

Kapur, J. N. Ballistic effects of bore resistance. Proc. Nat. Inst. Sci. India. Part A. **23** (1957), 395–404.

The effects of bore resistance have been handled by Corner for a standard quadratic form function by a weighing-factor method. Tawakley [same Proc. **22** (1956), 105–128] extended Corner's method to the case of a composite charge of two standard burning-function grains. This author reexamines the problem for the general case of a charge of general form-function. Beside analytic formulas he obtains 16 qualitative theorems, many of which state plausibly expected relations.

A. A. Bennett (Providence, R.I.).

Kapur, J. N. The internal ballistics of a tapered-bore gun. Proc. Nat. Inst. Sci. India. Part A. **23** (1957), 438–467.

Corner [Theory of the interior ballistics of guns, Wiley, New York, 1950; MR **12**, 213] concluded that for an isothermal model, the equations of internal ballistics for a tapered-bore gun remain of essentially the same form as for the case of uniform bore. However, this paper points out the need for the thorough reexamination here furnished. Covolume corrections are ignored. In the most general case his solutions are expressed in terms of the solutions of ordinary non-linear differential equations of the second order, and supplementary quadratures. The isothermal model with constant burning surface is explored analytically for three different types of tapered bores. Comparisons are made in three types of cases with results for uniform bore. Finally the theory is extended to the use of composite charges. The author concludes that "for the same maximum pressure and same chamber volume, the tapered-bore would give slightly better exterior ballistics." However, there remain questions incident to the rapid deformation of the projectile, Lagrange's correction, and strong bore resistance.

A. A. Bennett (Providence, R.I.).

Quantum Mechanics

Petiau, Gérard. Sur des fonctions d'ondes d'un type nouveau, solutions d'équations non linéaires généralisant l'équation des ondes de la Mécanique ondulatoire. C. R. Acad. Sci. Paris **244** (1957), 1890–1893.

En considérant les fonctions elliptiques de Jacobi $\mathrm{sn}(u, k)$, $\mathrm{cn}(u, k)$, $\mathrm{dn}(u, k)$ et certaines de leurs combinaisons classiques comme des généralisations des fonctions circulaires $\sin u$, $\cos u$, l'auteur propose diverses généra-

lisations de l'équation d'ondes relativistes de Schroedinger (ou Gordon-Klein) pour la particule libre sans spin, à savoir:

$$(\alpha) \qquad [\Box+\mu_1{}^2-\mu_2{}^2\psi^2]\psi=0,$$

$$(\beta) \qquad [\Box+\mu_1{}^2+\mu_2{}^2\psi^2]\psi=0,$$

$$(\gamma) \qquad [\Box-\mu_1+\mu_2{}^2\psi^2]\psi=0.$$

Ces équations admettent des solutions de type "ondes planes"

$$\psi=\lambda F\left[\frac{1}{\hbar'}(Wt-\mathbf{p}\cdot\mathbf{x}), k\right]$$

avec F=sn, cd pour (α); cn, $k$sd pour (β); dn, knd, cn, k'sd pour (γ); où $\hbar'=h/4K$, $4K$ étant la période réelle des fonctions de Jacobi et k, μ_1 et μ_2 des fonctions des constantes λ (arbitraire) et $\mu_0=m_0c/\hbar'$; m_0, W et $\mathbf{p}$ sont reliés par $(W/c)^2-(\mathbf{p}\cdot\mathbf{p})=m_0c^2$, m_0 étant la masse de repos de la particule. *D. Rivier* (Lausanne).

Petiau, Gérard. Sur la représentation des corpuscules élémentaires par des fonctions d'ondes solutions d'équations aux dérivées partielles non linéaires. C. R. Acad. Sci. Paris **244** (1957), 2580–2583.

Poursuivant l'analysée ci-dessus l'auteur cherche des généralisations de l'équation d'onde relativiste de Schroedinger (ou de Gordon-Klein) pour un corpuscule libre.

Posant les conditions suivantes: a) certaines solutions doivent être uniformes et bornées et du type "ondes planes", c.à.d. fonction de l'invariant τ; $\tau=(Wt-(\mathbf{p}\cdot\mathbf{x}))/\hbar$, b) certaines solutions doivent être fonctions à points critiques fixes de l'invariant u; $u^2=c^2t^2-(\mathbf{x}\cdot\mathbf{x})$; il est possible d'obtenir toutes les équations différentielles qui y satisfont grâce au mémoire de Painlevé [Acta Math. **25** (1902), 1–85]. L'auteur signale notamment l'équation

$$\Box\psi-\left(1-\frac{1}{n}\right)\left[\left(\frac{1}{c}\frac{\partial}{\partial t}\psi\right)^2-\Sigma\left(\frac{\partial\psi}{\partial x}\right)^2\right]\frac{1}{\psi}+\mu_0{}^2\psi=0$$

(n entier), qui permet d'introduire sans autre une quantification de la masse. *D. Rivier* (Lausanne).

Merat, Parviz. Sur la symétrie entre le retournement d'espace et le retournement du temps dans le formalisme spinoriel. C. R. Acad. Sci. Paris **244** (1957), 3036–3038.

Instead ofre presenting the extended Lorentz group as a linear group in a four-dimensional complex affine space, the author proposes to use a linear group in an eight-dimensional spin space obtained by considering four two-component spin spaces simultaneously. The author states erroneously that the transformation $x^{\sigma*}=-x^{\sigma}$ can only be represented by anti-linear transformations in the four-component spin space made up by taking two two-component spinors. *A. H. Taub* (Urbana, Ill.).

Merat, Parviz. Sur une généralisation de l'équation de Dirac pour les fermions lourds. C. R. Acad. Sci. Paris **244** (1957), 3131–3133.

The author has previously introduced an eight dimensional spin-space [see the preceeding review]. He now assumes that these eight component spinors satisfy a linear wave equation similar to the Dirac equation. The spinors are assumed to be functions of seven variables, the four coordinates of an event in space-time, and the three coordinates of a point in isotopic space. *A. H. Taub* (Urbana, Ill.).

Merat, Parviz. Sur une généralisation de l'équation de Dirac pour les fermions lourds en interactions avec des champs mésiques. C. R. Acad. Sci. Paris **245** (1957), 38–40.

In the paper reviewed above the author introduced a generalisation of the Dirac equation involving eight component spinors. In this paper he replaces the isotopic momenta $P_\alpha=(\hbar/i)\partial/\partial_{x\alpha}$ by $P_\alpha-(f/c)\phi_\alpha$ and discusses the resulting equation in the nonrelativistic approximation. *A. H. Taub.*

Hillion, Pierre. Limite non relativiste de la représentation hydrodynamique de l'équation de Dirac. C. R. Acad. Sci. Paris **245** (1957), 1394–1396.

It is shown that in the non-relativistic limit the tensor equations equivalent to the Dirac equations called the hydrodynamic representation of the Dirac equations reduce to the corresponding equations obtained from the Pauli equations for an electron with spin. *A. H. Taub.*

Hillion, Pierre. Interprétation causale de la théorie de l'atome d'hydrogène à l'aide de la représentation hydrodynamique de l'équation de Pauli. C. R. Acad. Sci. Paris **245** (1957), 1595–1597.

The author proposes to replace the description of a stationary regime, in the causal interpretation of quantum mechanics described by the hydrodynamical equations associated with the Pauli equations, by the requirement that the averages of various time derivatives vanish instead of the derivatives themselves vanishing. *A. H. Taub* (Urbana, Ill.).

Hubbard, J. The description of collective motions in terms of many-body perturbation theory. II. The correlation energy of a free-electron gas. Proc. Roy. Soc. London. Ser. A. **243** (1957), 336–352.

The theory developed in a previous paper [same Proc. **240** (1957), 539–560; MR **19**, 362] is applied to calculate the correlation energy of a free-electron gas. The theory involves no cut-off and gives a uniform description of collective motion effects in the long range limit and of particle motion effects in the short range limit. It is shown that in the lowest order the theory agrees with Bohm and Pines's plasma oscillation theory in the long-range limit, but is inadequate in the short-range limit. The theory is approximately evaluated to the next order, which is correct in the short-range limit, and is applied to calculate the correlation energy at several gas densities; the results are in good agreement with those of Bohm and Pines. (Author's summary.) *F. Rohrlich.*

Khuri, N. N.; and Treiman, S. B. Dispersion relations for Dirac potential scattering. Phys. Rev. (2) **109** (1958), 198–208.

The first author [Phys. Rev. (2) **107** (1957), 1148–1156] had previously shown that it was possible to derive dispersion relations for non-relativistic potential scattering, without the explicit use of microscopic causality. The derivation is here extended to the case of the scattering of a relativistic Dirac particle from a static potential, and it is shown that dispersion relations exist for this case, for a wide variety of potentials. Thus the evidence is increased that microscopic causality is an unsatisfactory means of identifying theories for which dispersion relations exist. *H. W. Lewis.*

Möglich, Friedrich. Zur Hydrodynamik eines Elektronengases. S.-B. Deutsch. Akad. Wiss. Berlin. Kl. Math. Phys. Tech. **1956**, no. 1, 29 pp. (1957).

Attempted justification of hydrodynamical methods in the collective description of electrons. Begins with a detailed derivation of the law that vortices of generalized momentum (kinetic momentum plus potential momentum proportional to the magnetic vector potential) cannot be created in an inviscid charged fluid [for this law see Gabor, Proc. I.R.E. **33** (1945), 792–805; Buneman, Proc. Phys. Soc. Sect. B. **63** (1950), 278–288]. A derivation of the Bernoulli equation for non-vortical motion follows (see also Buneman, loc. cit.).

It is shown that the action function W and the density n are canonical variables [see also Butcher, Phil. Mag. (7) **44** (1953), 971–979; MR **15**, 919], and that they are mutually conjugate. A function ψ, defined as $n^{\frac{1}{2}}\exp(iW/\hbar)$, is then proved to possess properties similar to a quantum-mechanical single-particle wave function; in particular, the current is obtained from it by the formula that furnishes the probability current in terms of the wave function. Also, ψ obeys a Schroedinger equation, with a potential energy which, in this paper, remains undetermined. The analogies between ψ and a wave function are claimed to lend support to the use of hydrodynamical methods in the collective description of electrons.

These attempts to relate non-vortical classical flow to the Schroedinger equation are essentially those of Madelung [Z. Physik **40** (1926), 322–326], Schönberg [Suppl. Nuov. Cim. **6**, 249 (1949)], Bohm [Phys. Rev. (2) **85** (1952), 166–179, 180–193; MR **13**, 709, 710] and Takabayasi [Nuovo Cimento (10) **3** (1956), 233–241; MR **18**, 173], but no reference is made to the extensive literature on this subject, nor is the comparison between classical and quantum calculations taken far enough to reveal the discrepancy in the potentials by Bohm's "quantum mechanical pressure", proportional to $n^{-\frac{1}{2}}\nabla^2 n^{\frac{1}{2}}$. Likewise, there is no reference to F. London's work on connecting irrotational (superconducting) flow with the multiparticle Schroedinger equation [F. London, Superfluids, v. 1, Wiley, New York, 1950], or to Feynman's [Progr. Low Temperature Phys. **1** (1955), 17–53] similar work on superfluidity. *O. Buneman* (Stanford, Calif.).

Belinfante, F. J.; Caplan, D. I.; and Kennedy, W. L. Quantization of the interacting fields of electrons, electromagnetism, and gravity. Rev. Mod. Phys. **29** (1957), 518–546.

This extremely long and laborious paper deals with the quantization of the gravitational field and its interaction with the electron and the photon fields. Throughout the paper the authors have freely borrowed from the work of the reviewer [S. N. Gupta, Proc. Phys. Soc. Sect. A. **65** (1952), 161–169, 608–619; MR **13**, 804; **14**, 417], but at places they have used some new devices of Dirac [P. A. M. Dirac, Canad. J. Math. **2** (1950), 129–148; MR **13**, 306] instead of the standard techniques of field theory. The resulting formalism is rather unintelligible, and it is not clear how it can be applied to physical problems.

S. N. Gupta (Detroit, Mich.).

Ladyženskaya, O. A. On the principle of limit amplitude. Uspehi Mat. Nauk (N.S.) **12** (1957), no. 3(75), 161–164. (Russian)

Let $L=-\Delta+c$ be the Schrödinger operator in R^3 with c vanishing outside a compact set. Consider the equation (*) $(L-l)u=f$. The problem is to single out a certain "physically reasonable" solution among the manifold of solutions of (*). One way is as follows (principle of limit amplitude): Let v be the solution of $v_{tt}+Lv=f\exp(il^{\frac{1}{2}}t)$ with zero Cauchy data. Consider the limit $u=\lim_{t\to\infty} v\exp(-il^{\frac{1}{2}}t)$. It is proved in the case f vanishing outside a compact set that this limit always exists and that u satisfies (*) and Sommerfeld's radiation condition. [Cf. Povzner, Mat. Sb. N.S. **32(74)** (1953), 109–156; MR **14**, 755.] *L. Gårding* (Lund).

Franke, Herbert W. Der Spannungstensor in der Wellenmechanik. Ann. Physik (6) **15** (1955), 155–156.

With the introduction by means of $\psi=\rho^{\frac{1}{2}}\exp(iw/\hbar)$ of a density ρ corresponding to the Schrödinger wave-function ψ one may derive a hydrodynamical pressure, and correspondingly a symmetrical stress tensor. Equations of centre of mass motion and of rate of increase of angular momentum are derived and also a form of virial theorem. *C. Strachan* (Aberdeen).

Brulin, O.; and Hjalmars, S. Relativistic wave equations for spin-2 particles with unique mass. Phys. Rev. (2) **107** (1957), 1730–1731.

Ohmura, Takashi. Extensions of variational methods. III. Determination of potential from phase shift function. Progr. Theoret. Phys. **16** (1956), 231–243.

Medvedev, B. V. On the construction of the scattering matrix. II. The theory with non-local interaction. Soviet Physics. JETP **5** (1957), 48–57.

N. N. Bogoljubov's method for constructing the scattering matrix is generalized to the case of a theory with non-local interactions. For such a theory, a scattering matrix is constructed which satisfies the physically necessary requirements. (From the author's summary.)

G. Källén (Copenhagen).

Case, K. M. Composite particles of zero mass. Phys. Rev. (2) **106** (1957), 1316–1320.

The author investigates some of the implications of the two-component theory of the neutrino [T. D. Lee and C. N. Yang, Phys. Rev. (2) **105** (1957), 1671–1675] for the construction of zero-mass particles of integral spin. It is shown that (i) one spin-zero field can be formed (involving the neutrino field bilinearly) which is a superposition of equal amounts of a scalar and a pseudoscalar field; (ii) field operators for particles of spin ms, where s is the spin of the two-component field, cannot be formed out of expressions containing m neutrino operators (m even). With $s=\frac{1}{2}$ and $m=2$ and 4, respectively, this therefore rules out the possibility of a two-component neutrino theory of light which is bilinear in the neutrino field and a two-component neutrino theory of gravitation which is quadrilinear in the neutrino field. *D. Feldman.*

Knight, James M.; and Toll, John S. Causality and the dispersion relation: S-matrix for Klein-Gordon and Dirac fields. Ann. Physics **3** (1958), 49–66.

If a wave packet is scattered by a spherically symmetrical scatterer, no scattered waves can appear before the wave packet has reached the scatterer. The application of this causality condition to relativistic particles is complicated by the fact that the incident waves can contain no frequencies between mc^2 and $-mc^2$. This difficulty is here overcome by an appeal to the possibility of bound states of the source, which do contain energies in this

'unphysical region'. It is then possible to derive the usual analytic behavior of the S-matrix in one-half of the complex frequency plane. If one assumes, in addition, that in the unphysical region S is real and bounded, apart from poles corresponding to bound states, one can write a Kramers-Kronig relation ('dispersion relation') connecting real and imaginary parts of S on the real axis. A similar relation is derived for the scattering amplitude, which may be translated back into relations connecting the matrix elements S_l for different multipoles l. It is hinted that these relations may be used to calculate the higher S_l when S_0 is known. *N. G. van Kampen* (New York, N.Y.).

★**Hartree, Douglas R. The calculation of atomic structures.** John Wiley & Sons, Inc., New York; Chapman & Hall, Ltd., London, 1957. xiii + 181 pp. $5.00.

The study of the electronic structure of the atoms has so far been based mainly on the "self-consistent-field" method. The atomic scientists are today grateful that, shortly before his death, the late D. R. Hartree succeeded in concluding a book giving an almost complete survey over this field, to which he and his father made such important contributions.

The book starts with an introductory chapter about the one-particle-approximation for treating many-electron systems. The solution of the Schrödinger equation by means of the variation principle is then discussed and, after a detailed analysis of the total energy of configurations consisting of closed shells, the self-consistent-field equations with and without exchange are derived. The numerical procedures used so far for solving these equations are fully described. Even the case of incomplete shells is treated. The important problem of deriving good starting functions is handled in a chapter about the variation of wave functions and fields with the atomic number. The next section is devoted to an investigation of the total energy and, finally, "better approximations" are briefly discussed.

The tables of mean radii, screening numbers, and reduced radial wave functions at the end of the book are quite useful. Perhaps the reader would expect to find a complete list of references to original self-consistent-field calculations in a book of this type, but the author refers instead to a previous review article in Rep. Progr. Phys. 11 (1948), 113–143 [MR 12, 210] and gives only a list of addenda bringing the bibliography up to date. This slight disadvantage could perhaps be removed by the publisher before the next printing of this excellent book, which will certainly become the standard text on atomic self-consistent-field calculations. *P.-O. Löwdin* (Uppsala).

Froese, Charlotte. The limiting behaviour of atomic wave functions for large atomic number. Proc. Roy. Soc. London. Ser. A. 239 (1957), 311–319.

In calculations of the electronic structure of atoms by means of the self-consistent-field (SCF) method, it is of great importance to get as good starting functions as possible. One way is to analyze the SCF-functions available and to derive auxiliary functions which are slowly varying with the atomic number N and which hence can be used for interpolation or extrapolation purposes. The purpose of the present paper is to investigate the "screening numbers" $\sigma(nl)$ in Hartree's reduced wave functions [Proc. Cambridge Philos. Soc. **51** (1955), 684–692], particularly in the mathematical limit $N\to\infty$. The radial wave functions are represented by series in $1/N$ where the coefficients are functions of r: the first coefficient is hydrogen-like, whereas the second is evaluated from the self-consistent-field equations by means of an electronic computer. Configurations including up to 28 electrons are treated, and the tables presented make it possible to determine starting functions for atoms with large N by a process of interpolation instead of extrapolation. *P.-O. Löwdin* (Uppsala).

Carlson, B. C.; and Keller, Joseph M. Orthogonalization procedures and the localization of Wannier functions. Phys. Rev. (2) **105** (1957), 102–103.

The method of "symmetric orthonormalization" is shown to have a least-squares property: it constructs those unique orthonormal functions which minimize the sum of squared distances (in Hilbert space) between each initial function and a corresponding function of the orthonormal set. The localization of Wannier functions is a consequence of this property, since they can be obtained from localized atomic orbitals by symmetric orthonormalization. The theorem further implies an optimal resemblance of Wannier functions to atomic orbitals. (Authors' abstract.) *P.-O. Löwdin* (Uppsala).

Simpson, William T. Modification of the vector model to include a first-order nonorthogonality correction. J. Chem. Phys. **25** (1956), 1124–1127.

When the vector model is used to describe chemical bonding it appears that the orbitals participating in bond formation are required to be orthogonal. The success of the model on an empirical basis is puzzling because chemical bonding will not occur without overlap. In the present work the many-electron functions, consisting of products of appropriately overlapping ordinary atomic orbitals, are orthogonalized. The characteristic simple form of the model is then retained provided that the exchange integral is replaced by a corrected exchange integral called a bond integral. The success of the model becomes less puzzling because this integral is usually evaluated empirically. The nature of the bond integral is examined in the framework of the Heitler-London treatment of the hydrogen molecule. (Author's abstract.) *P.-O. Löwdin* (Uppsala).

Araújo, J. M. Collective motions of a shell structure. Nuclear Phys. **1** (1956), 259–277.

The nucleus is considered to be described by an infinitely deep square well potential, the boundary of which is a function of time. The nucleon wave functions are taken to satisfy the time dependent Schroedinger equation for a free particle but with the boundary condition that they vanish at the well boundary. These wave functions are calculated approximately, and by means of them the expectation values of the energy (with neglect of Coulomb and spin-orbit interaction) and angular momentum are determined. These are found to be of the same form as in the case of the liquid drop model. The condition that the energy expectation value be independent of the time leads to an eigenfrequency equation. The calculated frequencies and static deformabilities show considerable fluctuations. Some experimental evidence for these fluctuations is adduced. *N. Rosen.*

Skyrme, T. H. R. Nuclear moments of inertia. Proc. Phys. Soc. Sect. A. **70** (1957), 433–444.

Using his earlier work [Proc. Roy. Soc. London. Ser. A. **239** (1957), 399–412; MR **19**, 96] for relating a phenom-

enological collective model of the nucleus to a full description, the author develops a variation principle for calculation of the moment of inertia $\mathcal{I}$ of a rotating nucleus as it occurs in the energy formula $E=(\hbar^2/2\mathcal{I})I(I+1)$. The wave function occurring in the variation principle involves nucleonic coordinates and collective coordinates. This may be made to depend only on a function of the former which is taken to be a determinant of single-particle wave-functions, conveniently for a deformed oscillator well. An order of magnitude of rigid $\mathcal{I}/\mathcal{I}_{\text{rigid}}$ is derived, but, in this difficult problem, so many lines of approximation are used that the reviewer finds it difficult to estimate the validity of the treatment.
C. Strachan (Aberdeen).

Kresnin, A. A.; and Rozentsveig, L. N. Polarization effects in the scattering of electrons and positrons by electrons. Soviet Physics. JETP **5** (1957), 288–292.

Using standard methods, a formula is obtained giving the cross sections for the scattering of a polarised beam of electrons and of positrons by a polarised target. The difference from that for scattering by an unpolarised target arises in second order. *A. Dalgarno* (Belfast).

Preuss, H. Zur "Methode der Atome in Molekülen". Z. Naturf. **12a** (1957), 599–603.

The method of atoms in molecules is generalized in such a way as to incorporate the correct asymptotic behaviour of molecular energy curves both for infinite nuclear separation (the isolated atoms) and for zero nuclear separation (the united atom). The systems H_2^+ and H_2 are treated in detail and quite accurate results are obtained with relatively crude wave functions.
A. C. Hurley (Melbourne).

Duculot, Camille. Méthode de détermination directe des représentations irréductibles associées aux harmoniques et combinaisons de vibrations moléculaires. Acad. Roy. Belg. Cl. Sci. Mém. Coll. in 8° **30** (1957), no. 4, 90 pp.

The irreducible representations associated with harmonics and combinations of molecular vibrations are usually derived by analysing direct product representations by means of the character table of the group. The author develops a simpler direct method for solving this problem, which does not use the character table. The three dimensional irreducible representations of the tetrahedral and octahedral groups are related to two- and one-dimensional representations of subgroups. When products of representations are expanded in this way the resulting terms may be recombined to give representations of the full group. For the nth harmonic, this technique leads to explicit formulae for the multiplicities of the various irreducible representations in terms of n.
A. C. Hurley (Melbourne).

Wilson, E. Bright, Jr. Analysis of spin-spin interaction in the nuclear magnetic resonance spectra of symmetrical molecules. J. Chem. Phys. **27** (1957), 60–68.

Molecular symmetry is used to simplify calculations of nuclear spin-spin coupling in molecules in the liquid state. The application of group theory to this problem is similar to that for the problem of molecular vibrations. Symtrifluorobenzene is used to illustrate the general theory.
A. C. Hurley (Melbourne).

Handler, George S.; and Arnold, James R. Use of polar coordinates for molecular wave functions. J. Chem. Phys. **27** (1957), 144–146.

Wave functions for H_2, H_3^+ linear and H_3^+ triangular are expanded in terms of the polar coordinates of a single centre. For this system of coordinates the basic integrals may be evaluated very simply. However, the convergence of the expansion is found to be so slow that the method is not generally useful. *A. C. Hurley* (Melbourne).

Hillion, Pierre. Solution explicite de la représentation hydrodynamique de l'équation de Pauli dans le cas de l'atome d'hydrogène. C. R. Acad. Sci. Paris **245** (1957), 1516–1518.

Tzou, Kuo-Hsien. Formulation matricielle du champ vectoriel et du champ de spin maximum 1 et comparaison de leur structure interne. C. R. Acad. Sci. Paris **244** (1957), 2137–2140.

Hue, Jean; et Seiden, Joseph. Transitions par irradiation indirecte. C. R. Acad. Sci. Paris **244** (1957), 2157–2160.

Omnès, Roland. Sur les théories des champs invariantes par rapport au groupe conforme. C. R. Acad. Sci. Paris **245** (1957), 1382–1384.

The author lists various properties of the generators of the conformal group of space-time and states that, in view of these properties, the conformal group does not form a framework in which elementary particles can be described. *A. H. Taub* (Urbana, Ill.).

Klauder, John; and Wheeler, John A. On the question of a neutrino analog to electric charge. Rev. Mod. Phys. **29** (1957), 516–517.

The two-component neutrino equation in a multiply-connected (topologically toroidal) Riemannian space with time-independent metric is shown to admit of no zero-frequency solutions. Since such solutions would correspond to the notion of "charge" for the Maxwell equations in this space, it is concluded that "there is no reason to expect for the neutrino field a concept of charge having any direct analogy to electric charge". *S. Deser.*

Reyntjens, Jacques. Etude théorique de la création de paires dans le champ d'un électron. Ann. Soc. Sci. Bruxelles. Sér. I. **71** (1957), 189–203.

The authors calculate the cross section for the production of an electron-positron pair by a photon in the vicinity of an electron. The paper lacks originality.
S. N. Gupta (Detroit, Mich.).

Dell'Antonio, G. F.; and Duimio, F. On the relation between the Lee model and ordinary meson theory. Nuovo Cimento (10) **5** (1957), 1636–1645.

In order to examine whether the Lee model [Phys. Rev. (2). **95** (1954), 1329–1334; MR **16**, 317] may be regarded as an approximation to neutral scalar theory, an energy cut-off is used which eliminates states in which the total energy of mesons is greater than a fixed value, and which, therefore, because of rest mass, limits the number of mesons. This might be regarded as an extended Lee model. Only states which belong to the zero eigenvalue of one constant of motion are termed "physical". Usual renormalization procedures are used.

Particular examples are the Lee model which leads to inconsistencies, and those in which the upper limit of allowed meson energies is in steps equal to the meson rest mass. (There appears to be a misprint in equation (1).)
C. Strachan (Aberdeen).

Migdal, A. B. The momentum distribution of interacting Fermi particles. Soviet Physics. JETP **5** (1957), 333–334.

For a system of a large number of fermions in interaction, p_0 denotes the momentum of the top particle and $n(p)=\langle a_p^+ a_p\rangle$ defines the momentum distribution. The author shows that a result of H. Lehmann [Nuovo Cimento (9) **11** (1954), 342–357; MR **17**, 332] implies that $0<|n(p_0-0)-n(p_0+0)|\leq 1$, which implies that the momentum distribution of the system has a discontinuity at $p=p_0$. *A. J. Coleman* (Toronto, Ont.).

Tati, Takao. A theory of elementary particles. Progr. Theoret. Phys. **18** (1957), 235–246.

The author considers the framework of quantum field theory in the circumstances when one gives up the concept of a rigid space-time and replaces it by a fluctuating space-time. The usual kinematics of quantum theory (state vectors, expectation values, etc.) is retained. Annihilation and creation operators in momentum space are introduced, the momentum label representing the momentum of bare particles. These field operators can be fourier analyzed. However, the x^μ of the fourier analysis do not represent space-time coordinates in this theory. While the latter do not appear directly, the concept of order of measurements is assumed still to hold. If O_1, O_2, $\cdots$ represents the ordering parameter, then the development of the state vector is postulated to be $\psi_2=\exp[iP_\mu C^\mu]\psi_1$, where P_μ is the energy-momentum vector of the system and $C^\mu(O_2O_1)$ is an undetermined four-vector which replaces the usual $x_2^\mu-x_1^\mu$ of the Schroedinger equation. The analogues of rods and clocks are obtained by introducing systems of many degrees of freedom. If one picks four observables Ω^μ for such systems, the measurement of the expectation values of Ω^μ determine the mean values of C^μ, i.e., $\bar{C}^\mu$. In general, the relation $\bar{C}^\mu(O_1O_2)+\bar{C}^\mu(O_2O_3)=\bar{C}^\mu(O_1O_3)$ does not hold due to quantum fluctuations. (If the above transitive relation were valid, then $\bar{C}^\mu(O_1O_2)=x_1^\mu-x_2^\mu$ and the theory would reduce down to conventional field theory in a rigid space time). The relation between the author's theory and non-local field theories is briefly discussed.
R. Arnowitt (Syracuse, N.Y.).

Ioffe, B. L.; Okun', L. B.; and Rudik, A. P. The problem of parity non-conservation in weak interactions. Soviet Physics. JETP **5** (1957), 328–330.

Some cases are discussed in which parity non-conservation is not observable unless charge conjugation invariance is also absent. *J. C. Taylor* (London).

Heisenberg, W. Quantum theory of fields and elementary particles. Rev. Mod. Phys. **29** (1957), 269–278.

This is a survey of the work done by the author and his associates [R. Ascoli and W. Heisenberg, Z. Naturf. **12a** (1957), 177–187; MR **19**, 99] on a model for a theory of elementary particles. It is based on the non-linear spinor equation

$$\gamma\cdot\partial\psi-l^2\psi(\psi^+\psi)=0,$$

where l is a constant length. This equation is, it is argued, inconsistent with the usual commutation relations. These are altered to meet the difficulty by introducing a "dipole ghost" state, one of whose two components has a negative normalization in Hilbert space. This innovation, it is claimed, does not destroy the unitarity of the S-matrix.

The equations are solved approximately by a sort of Tamm-Dancoff method. A number of eigenstates are found corresponding to fermions and bosons with masses of the order of l^{-1}. The theory is shown to have features of exactly the same structure as electromagnetism, with a fine structure constant calculated to be of order 10^{-2}. This effect is connected with the new form of the commutation relations.

Possible extensions of the theory to include isotopic spin and weak interactions are discussed. *J. C. Taylor.*

Feynman, R. P.; and Gell-Mann, M. Theory of the Fermi interaction. Phys. Rev. (2) **109** (1958), 193–198.

The Dirac equation for spin $\frac{1}{2}$ fields is equivalent to a second order equation for a two-component spinor. So it is not unnatural that the Hamiltonian for Fermi interactions should be written in terms of these two-component spinors without derivatives (although no free-field Hamiltonian in this form is given). Together with the experimental requirements that neutrino, electron, negative muon and nucleons are all 'particles' (i.e. are represented by similar two-component spinors), these considerations suggest a unique form of universal Fermi interaction ($V-A$, with maximum parity non-conservation). At the time of publication, this theory was in apparent contradiction only with He^6 recoil experiments and the branching ratio in pion decay.

Assuming this form of interaction, the coupling constants found for O^{14} and for muon decay are very nearly identical. This fact is explicable if the vector part of the nucleon contribution to the interaction is a conserved current, constructed out of nucleons and mesons, like the ordinary electric current. This hypothesis predicts the decay $\pi^-\to\pi^0+e+\bar{\nu}$ with the same ft value as O^{14}.

If hyperons are included in the universal Fermi interaction, strange particle decay with parity non-conservation is to be expected. Hyperons should also occasionally undergo beta decay. *J. C. Taylor* (London).

Hanawa, Sigeo. Mathematical formulation of the Gell-Mann-Nishijima scheme for new particles. Progr. Theoret. Phys. **17** (1957), 592–602.

Good, I. J. On the masses of the proton, neutron and hyperons. J. Roy. Naval Sci. Service **12** (1957), 144.

See also: Numerical Methods: Clendenin. **Structure of Matter:** Price; Lundqvist. **Relativity:** Deser.

Relativity

Builder, G. The resolution of the clock paradox. Austral. J. Phys. **10** (1957), 246–262.

The author argues in detail that the clock paradox can be resolved completely in terms of the restricted or special theory of relativity, without recourse to the general theory. He claims that the appearance of paradox arises through an incorrect application of the Lorentz transfor-

mations, in which a calculation is first made of the 'rate' of a moving clock and this 'rate' is then applied without due regard to its exact meaning. The asymmetry of the predicted retardation is regarded as the natural concomitant of the dynamical asymmetry associated with the fact that one clock is subjected to acceleration and the other not. The author criticizes the treatment of the paradox by means of the principle of equivalence of general relativity as given by R. C. Tolman, [Relativity, thermodynamics and cosmology, Oxford, 1934, pp. 194–197]. *G. J. Whitrow* (London).

Dingle, H. The resolution of the clock paradox. Austral. J. Phys. **10** (1957), 418–423.

The author makes a detailed reply to the arguments set out in the paper reviewed above. He argues that it is illegitimate to suppose that only one of the two clocks concerned experiences acceleration. Instead, the principle of relativity, which Dingle interprets as "the principle that nature allows of no criterion for deciding which of two relatively moving bodies is the 'moving' one", necessarily requires, in his view, that the clocks in question shall agree on reunion. *G. J. Whitrow* (London).

Infeld, L. On the Lagrangian in special relativity theory. Bull. Acad. Polon. Sci. Cl. III. **5** (1957), 491–495, XLI. (Russian summary)

Following Szamosi [Acta Phys. Acad. Sci. Hungar. **6** (1956), 207–215; MR **18**, 445], the author adds to a Lorentz-invariant particle Lagrangian a term representing the condition that the 4-velocity be a unit vector, and studies the consequences of this for some simple examples. *F. A. E. Pirani* (London).

Halbwachs, Francis. Sur un cas particulier du mouvement de la goutte de Bohm et Vigier. C. R. Acad. Sci. Paris **245** (1957), 1513–1516.

Bondi, H. Negative mass in general relativity. Rev. Mod. Phys. **29** (1957), 423–428.

In Newtonian mechanics, consider two gravitating particles, masses M_1, M_2, located at $z=h_1$, $z=h_2$, respectively, in a cylindrical coordinate-system (r, θ, z). Suppose that $h_2>h_1>0$ and assume that the forces acting on the particles have a potential

$$\text{(I)}\quad V=u^2\phi_0-GM_1(r^2+(z-h_1)^2)^{-\frac{1}{2}}-GM_2(r^2+(z-h_2)^2)^{-\frac{1}{2}},$$

where

$$\text{(II)}\quad \phi_0=\tfrac{1}{2}\log((r^2+z^2)^{\frac{1}{2}}+z),$$

u is an unknown constant velocity and G is the constant of graviation. Then the equations of motion in the z-direction of the particles are, respectively,

$$M_1\frac{d^2h_1}{dT^2}=-M_1u^2\left(\frac{1}{2h_1}-\frac{GM_2}{u^2}\frac{1}{(h_2-h_1)^2}\right),$$

$$M_2\frac{d^2h_2}{dT^2}=-M_2u^2\left(\frac{1}{2h_2}+\frac{GM_1}{u^2}\frac{1}{(h_2-h_1)^2}\right).$$

Thus if the particles are to remain at relative rest

$$\text{(III)}\quad \frac{GM_2}{u^2}\frac{1}{(h_2-h_1)^2}=\frac{1}{2h_1},\qquad \frac{GM_1}{u^2}\frac{1}{(h_2-h_1)^2}=-\frac{1}{2h_2},$$

and therefore M_1 must be negative and M_2, positive. Objections to this treatment are (a) the lack of physical significance for the part $u^2\phi_0$ of the potential V. It introduces a singularity along $r=0$, $z\leqq 0$, one of whose manifestations is that M_2 tends to infinity as h_1 tends to zero. (b) The presence of the same G in both of the equations (III); the constant of gravitation is known to exist experimentally only for bodies with positive masses.

In his paper Bondi does not refer to the classical problem outlined above. He solves the corresponding problem in general relativity, using abstract units so that G and the velocity of light, c, do not appear explicitly in his analysis. For a space-time with cylindrical symmetry

$$\text{(IV)}\quad ds^2=e^{2\phi}dt^2-e^{-2\phi}(e^{2\sigma}(dr^2+dz^2)+r^2d\theta^2),$$

where ϕ, σ are functions of r, z alone. Since ϕ is known to satisfy Laplace's equation in empty space for this relativity problem also, we can choose $\phi=V/c^2$ and thus write the masses in length-units as $m_1=GM_1/c^2$, $m_2=GM_2/c^2$. The condition that the function σ should exist gives (Bondi (22) and (24)) the relations (III) with $u=c$. Bondi points out that the space-time of special relativity, viz:

$$\text{(V)}\quad ds^2=dt'^2-(dr'^2+dz'^2+r'^2d\theta^2)$$

can be transformed into a special case of (IV) wherein ϕ is ϕ_0 and

$$\sigma=\phi_0-\tfrac{1}{4}\log(4(r^2+z^2))$$

(Bondi (10) to (13) with $a=0$ as in (18) onwards). It can be shown that the coordinate system (t, r, θ, z) for this particular case of (IV) is an accelerated system relative to the inertial system (t', r', θ, z') of (V). With the two masses present, $\phi=V/c^2$, and the field at infinity reduces to the special relativity field (V) expressed, however, in the accelerated coordinate system. But the physical interpretation of this acceleration is still lacking, the singularity along $r=0$, $z\leqq 0$ is still there and no justification for the presence of the same G concealed in the definitions of $m_1(<0)$ and $m_2(>0)$ is forthcoming.

Bondi also determines the form of ϕ in the interiors of the "particles" though he does not calculate the corresponding σ's (Bondi, second formulae of (18), (19)). He also discusses some of the mathematical features of the restriction that his solution holds only for the region on the positive side of the plane $z=0$. *G. C. McVittie.*

Papapetrou, A. Eine neue Formulierung in der Relativitätstheorie. Schr. Forschungsinst. Math. **1** (1957), 210–221.

Après une discussion de plusieurs questions concernant les principes de la relativité générale, en particulier des différents points de vue concernant les systèmes de coordonnées privilégiés, l'auteur expose brièvement [pour les détails voir Papapetrou, Z. Physik **139** (1954), 518–532; MR **16**, 870] une théorie de la gravitation où il suppose a priori que la métrique a la forme

$$ds^2=-e^{-\varphi}(dx^2+dy^2+dz^2)+e^{\chi}dt^2.$$

Prenant pour Lagrangien

$$L=g^{\alpha\beta}\left(\Gamma^{\mu}_{\alpha\nu}\Gamma^{\nu}_{\beta\mu}-\Gamma^{\mu}_{\alpha\beta}\Gamma^{\nu}_{\mu\nu}\right)+\kappa g_{\mu\nu}C^{\mu\nu}$$

il obtient l'équation:

$$R_i^{\,i}-\frac{3}{2}R+\frac{d\chi}{d\varphi}(R_0^{\,0}-\tfrac{1}{2}R)=\kappa\left(C_i^{\,i}+\frac{d\chi}{d\varphi}C_0^{\,0}\right).$$

Il détermine χ comme fonction de φ par le principe d'équivalence de Von Laue et trouve $\chi=-\varphi$.

Il étudie ensuite les équations du mouvement (ce sont les mêmes qu'en relativité générale), la solution à symétrie sphérique (Schwarzschild en première approximation) et discute, dans cette théorie, le rôle de la transformation de Lorentz. *Y. Fourès-Bruhat.*

Trautman, A. On the conservation theorems and coordinate systems in general relativity. Bull. Acad. Polon. Sci. Cl. III. **5** (1957), 721–727, LXIII. (Russian summary)

The author attempts to develop a theory of conservation laws within the framework of the general theory of relativity by requiring that the conserved quantities and the conservation laws should be defined in such a manner that (a) the usual invariance conditions are satisfied; (b) the conservation laws be capable of an integral formulation; and (c) for flat space-time the conservation laws be reducible to the ten conservation laws of the special theory of relativity. It is remarked that the well-known equations $T^{ij}{}_{;j}=0$ do not satisfy conditions (b) and (c). The author's method depends on a variational principle and the application of isometries (i.e. point transformations generated by solutions of Killing's equations [L. P. Eisenhart, Riemannian geometry, Princeton, 1926, p. 234]). It is shown that to each isometry there corresponds a conservation law which fulfills the requirements (a), (b), (c). However, the author is aware of the fact that the number of such laws is equal to the number of independent isometries of the space-time continuum V_4, and that this number depends on the symmetry properties of V_4. In particular, V_4 admits 10 isometries if and only if it is a space of constant curvature. The author's method is applied also to the study of certain first integrals of the differential equations of the geodesics. The paper concludes with some remarks concerning preferred coordinate systems. *H. Rund.*

Davidson, W. General relativity and Mach's principle. Monthly Not. Roy. Astr. Soc. **117** (1957), 212–224.

A quantitative analysis of the role of Mach's principle (according to which inertia depends upon a kind of mutual action between bodies) from the point of view of general relativity was given by Einstein [The meaning of relativity, 4th ed., Princeton, 1953, p. 94 et seq.; MR **14**, 805]. On the basis of the same approximations the author of the present paper repeats these calculations in order to indicate an error of detail in the work of Einstein. In fact, the Maxwell-type pondermotive equation (of a weak field), which expresses the dependence of inert mass on ponderable masses, is derived, and while according to Einstein (loc. cit., p. 97, first of equations (118)) the inert mass is proportional to $1+\bar{\sigma}$, it is shown that this term should be replaced by $1+3\bar{\sigma}$, where $\bar{\sigma}=(\kappa/8\pi)\int(\sigma/r)dV_0$. However, the principal object of this paper is the presentation of a new theory which seeks to overcome certain difficulties raised by Sciama [Monthly Not. Roy. Astr. Soc. **113** (1953), 34–42; MR **15**, 566] in a tentative theory designed to provide a combination of Newton's laws of motion and of gravitation with inertial frames determined by Mach's principle. A novel feature of the author's approach is the fact that he defines the components of the 4-potential (which appears in the pondermotive equation of the field) in terms of the total $g_{\mu\nu}$ and not, as is customary in the quasi-Galilean case, by the deviation of the $g_{\mu\nu}$ from their Galilean values. This is consistent with the basic principle according to which one should strive to account for the whole of inertia in this manner. The results of the application of this theory to three examples illustrate the plausibility of this approach. It is indicated that a stationary cosmological solution appears to be particularly suitable from the point of view of a complete incorporation of Mach's principle in the general theory of relativity. The paper concludes with comparison with Sciama's theory. *H. Rund* (Durban).

Faivre-Blancheton, Éliane. Équations aux variations de la relativité générale. C. R. Acad. Sci. Paris **245** (1957), 284–286.

Let $ds_\varepsilon^2=g_{\alpha\beta}(x,\varepsilon)dx^\alpha dx^\beta$ be a family of space-time metrics depending on a parameter ε. Assume that $g_{\alpha\beta}=a_{\alpha\beta}+\varepsilon\gamma_{\alpha\beta}+O(\varepsilon^2)$. The author expands Einstein's field equations in powers of ε and finds the equations determining $\Gamma_\alpha{}^\beta=(\delta_\alpha{}^\gamma a^{\beta\delta}-\frac{1}{2}\delta_\alpha{}^\beta a^{\gamma\delta})\gamma_{\delta\gamma}$. *F. A. E. Pirani.*

Infeld, L. The equations of motion in general relativity theory and the action principle. Acta Phys. Polon. **16** (1957), 177–210. (Russian summary)

The problem of showing how the field equations of general relativity theory imply the equations of motion of material bodies has hitherto been dealt with from two alternative points of view: (i) matter is represented discontinuously as singularities of the solutions of the field equations $R_{ik}=0$; (ii) the field equations are taken as $R_{ik}-\frac{1}{2}g_{ik}R=-8\pi T_{ik}$, that is, matter is represented by a regular phenomenological tensor T_{ik}. The paper under review adopts a standpoint intermediate between these two, in that, though a phenomenological tensor T_{ik} is retained in the field equations, it is now taken to be singular from the start, the representation of the individual particles involving delta functions. Some very formal work involving suitably defined delta functions leads to Lagrangian equations for the world lines of p particles. The field equations are then solved and the equations of motion obtained in explicit form by an approximation procedure involving expansions in inverse powers of the velocity of light, the explicit calculations being carried out as far as the first post-Newtonian approximation. The question of coordinate conditions is discussed. [This paper is reprinted unchanged in Rev. Mod. Phys. **29** (1957), 398–411.] *H. A. Buchdahl.*

Ikeda, Mineo. On Einstein's relativistic theory of the non-symmetric field. Progr. Theoret. Phys. **18** (1957), 154–162.

The author considers that in the usual formulations of Einstein's asymmetric unitary field theory the construction of a transposition-invariant Lagrangian, or the introduction of U_{kl}^i, are artificial. He therefore proposes to alter the theory in such a way that its transposition-invariance is a consequence of the covariance of the field equations. For this purpose he replaces the fundamental tensor Δ-density $\mathfrak{g}^{kl}$ [using here the terminology of J. A. Schouten, Ricci-calculus, 2nd ed., Springer, Berlin, 1954, p. 12; MR **16**, 521] by the sum $\mathfrak{g}^{kl}$ of a symmetrical Δ-density $\mathfrak{g}^{\underline{kl}}$ and a skew-symmetric ordinary density $\mathfrak{g}^{\underset{\sim}{kl}}$. Then $\mathfrak{g}_{\underline{kl}}$ is a tensor, but $\mathfrak{g}_{\underset{\sim}{kl}}$ is a W-tensor; so that in a transformation of coordinates with negative Jacobian, $\mathfrak{g}_{kl}$ may be transformed as a tensor, provided that its subscripts are interchanged after this transformation. A similar result holds for the linear connection Γ_{kl}^i. Since the covariant derivative as usually defined does not now possess invariant meaning pseudo covariant derivatives have to be introduced. Thereafter field equations are

derived from a Lagrangian $\mathfrak{L}$ which is a scalar Δ-density. Since $\mathfrak{L}$ is a scalar made up of g_{kl}, Γ^t_{kl} and $\Gamma^t_{kl,i}$ alone, its transposition-invariance follows from a comparison of transformations with positive and negative Jacobian respectively. {It seems to the reviewer that the addition of ordinary and Δ-densities is itself most artificial. Moreover the usual equation

$$g_{kl,s}=\Gamma^t_{ks}g_{tl}+\Gamma^t_{ls}g_{kt}$$

is adopted here without comment; yet its form is determined only by its postulated transposition invariance.}

H. A. Buchdahl (Hobart).

Arcidiacono, Giuseppe. Sul significato fisico della "teoria di relatività finale". Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) **20** (1956), 463–469.

Continuing work of L. Fantappiè [same Rend. (8) **12** (1952), 285–290; **17** (1954), 158–165; **19** (1955) 367–373; MR **14**, 117; **17**, 795, 1162], the author discusses the group of de Sitter space (i.e., homogeneous real linear transformations leaving invariant a quadratic form in five space with four positive and one negative coefficient) from the viewpoint of observers in relativity theory and also in relation to the Klein-Gordon equation.

I. E. Segal (Chicago, Ill.).

Taub, A. H. Singular hypersurfaces in general relativity. Illinois J. Math. **1** (1957), 370–388.

It has been suggested that in general relativity the theory of a perfect fluid should allow for the existence of shock waves, and in the present paper conditions are found under which there may exist in space-time certain singular 3-dimensional hypersurfaces. These are hypersurfaces across which the energy tensor and the metric tensor and its derivative may have discontinuous components, and the relations holding between these components across the hypersurface are generalisations of the classical Rankine-Hugoniot equations. The generalisations previously given for special relativistic hydrodynamics now appear as special cases. The method used is an application and extension of a variational principle introduced by the author in a previous paper, and the conditions found to be satisfied by the discontinuous tensor components across a singular hypersurface are related to those given by other writers on the subject.

A. G. Walker (Liverpool).

Deser, S. General relativity and the divergence problem in quantum field theory. Rev. Mod. Phys. **29** (1957), 417–423.

Two arguments are advanced to suggest that "a quantum field theory including gravitational effects seems to rid itself of point-singularity difficulties". First, the usual proofs of the infinitude of renormalization constants depend upon properties of the total linear momentum of the fields which do not obtain in General Relativity Theory. Secondly, a discussion of the problem of quantizing a scalar matter field by means of Feynmann's method of integration of paths shows that the contribution to the singularity on the light cone comes from integration over a set of measure zero in the function space.

A. J. Coleman (Toronto, Ont.).

Power, Edwin A.; and Wheeler, John A. Thermal geons. Rev. Mod. Phys. **29** (1957), 480–495.

This paper extends the work of J. A. Wheeler on geons [Phys. Rev. (2) **97** (1955), 511–536; MR **16**, 756]. In the thermal geon, electromagnetic waves in a spherical shell are held together by their own gravitational field and have all possible modes of vibration. The latter field is assumed to be static and spherically symmetric, and each bound mode of circular frequency Ωc is assigned an energy $\hbar c\Omega[-1+\exp(\hbar c\Omega/T)]^{-1}$, where $\hbar$ is the quantum of angular momentum, and T is the temperature. The electromagnetic field is derived from Maxwell's equations, formed in the curved space-time of the gravitational field, and has electric, or magnetic multipole character. The mean stress-energy tensor at any part of the shell is determined for the given distribution of energy in the modes. Its value agrees with that for black body radiation in equilibrium in a gravitational field, when the latter is corrected for a solid angle factor. This stress-energy tensor, which itself depends on the metric tensor, is then used in Einstein's field equations to determine the gravitational field.

The properties of the thermal geon are specified by the one parameter T, and in this work the assumption is made that T is sufficiently small for classical analysis to be permissible, and for leakage of energy through the potential barrier to be neglected. The differential equations are written in a dimensionless form with T replaced by an eigenvalue parameter, the value of which determines the mass of a geon of temperature T. Numerical solution of the equations is carried out with the eigenvalue parameter adjusted so that the gravitational field in the shell joins smoothly onto the Schwarzschild field at the outer boundary and onto flat space-time at the inner boundary.

C. Gilbert (Newcastle-upon-Tyne).

Fourès-Bruhat, Yvonne. Problème des conditions initiales. C. R. Acad. Sci. Paris **245** (1957), 1384–1386.

Treder, H. Stromladungsdefinition und elektrische Kraft in der einheitlichen Feldtheorie. Ann. Physik (6) **19** (1956), 369–380 (1957).

The author proposes a new definition of charge-current in Einstein's "weak" non-symmetric unified theory [The meaning of relativity, 3rd ed., revised, Princeton, 1950, Appendix II; MR **14**, 97]. In the lowest approximation he obtains the Coulomb force between point charges, but also, unfortunately, an additional force independent of distance.

F. A. E. Pirani (London).

See also: Differential Geometry: MacDuffee. **Quantum Mechanics:** Merat; Brulin and Hjalmars.

Astronomy

Contopoulos, George. On the relative motions of stars in a galaxy. Stockholms Obs. Ann. **19** (1957), no. 10, 32 pp.

In the first part of this paper the relative motions of stars in the plane of symmetry of a galaxy are studied. The system of coordinates used is a rotating one, the angular velocity being the circular velocity at a given distance r_0 from the center. The coordinates are $\xi=r-r_0$, $\eta=r_0\theta$. Continuing the work of Lindblad and others, a second approximation is studied, which gives ξ as an elliptic function of the time. When the dimensions of the orbits are of the same order of magnitude as a spiral arm, a second approximation is considered to be good enough.

As to the form of orbits, their concavity and convexity towards the center is studied.

The second part deals with relative motions in three dimensions under the assumption that the field of the galaxy has an axis and a plane of symmetry. Comparison with planar motions is made. Projections of the orbits on the plane passing through the axis of symmetry and the moving point are considered. The real three dimensional orbit always lies inside a certain torus. The orbits are not periodic in general, nor are their projections on the rotating plane mentioned above. Recurrence and some other related properties of the orbits are discussed.

E. Leimanis (Vancouver, B.C.).

Minin, I. N. On spherical systems of gravitating bodies. Leningrad. Gos. Univ. Uč. Zap. 190. Ser. Mat. Nauk **29** (1957), 30–39. (Russian)

Masani, Alberto. La teoria delle stelle pulsanti. Rend. Sem. Mat. Fis. Milano **25** (1953–54), 41–72 (1955).

This is an expository article on the elements of a theory of adiabatic radial oscillation of the stars in acoustic approximation, restating such results as can be found, e.g., in Rosseland's "Pulsation theory of variable stars" [Oxford, 1949] and other standard sources. An outline of a comparison between theory and observation is also given, and the well-known discrepances are pointed out as requiring appropriate modifications of the basic theory.

Z. Kopal.

Mathur, V. Swarup. Equation of state of elements from the relativistic Thomas-Fermi theory. Case of extreme degeneracy. Proc. Nat. Inst. Sci. India. Part A. **23** (1957), 430–437.

The equation of state of matter at extremely high pressures but low temperature, of some astrophysical interest, is obtained from the relativistic Thomas-Fermi equation with fewer approximations than previous authors have sued.

P. W. Anderson.

Sen, D. K. A static cosmological model. Z. Physik **149** (1957), 311–323.

An attempt is made to interpret the red-shift phenomenon for the universe at large through Lyra's [Math. Z. **54** (1951), 52–64; MR **12**, 858] geometry in which a gauge factor for displacement vectors is employed in addition to the metrical tensor. The energy-tensor for a fluid material is defined in the usual way relative to the metric. Field equations are obtained by equating the energy-tensor to a geometrical tensor involving the usual Einstein tensor plus terms depending on the vector introduced by the gauging. No attempt is made to show that these field equations would conserve mass and momentum, as in general relativity; as far as this reviewer can discover, this conservation is abandoned. When it is assumed that the gauging vector has a constant time-like component (denoted by β) only, and that the metrical tensor is statical and spherically symmetric, the field equations show that the metrical tensor is that of the Einstein universe. If the pressure of the material is zero, a positive density is possible only if β is imaginary, say $\beta=2i\alpha$, in Sen's notation. The world-line of a particle is defined to be a curve whose tangential vector is transferred parallely to itself, allowing for the gauge factor. The world-line is not a geodesic of the metric. In the model universe he has obtained, Sen works out the world-line of a light-emitting particle P which has fixed spherical polar coordinates (r, θ, ϕ), so that $r=a$, say. The coordinates of points on the world-line are expressed in terms of a parameter S which need not, as Sen assumes in his equations (2.21) to (2.24), be identical with the proper-time s that is defined through the metric (2.4). The parameter S and the coordinate-time t are related by (Sen's (2.27))

$$\frac{dS}{dt} = (1-i\alpha S)^{-1}. \tag{1}$$

But for an Einstein universe metric (2.4), the proper-time s along a world-line for which (r, θ, ϕ) are constants, is related to t by

$$\frac{ds}{dt} = 1. \tag{2}$$

Hence, if $s=S$ as Sen assumes, (1) and (2) are consistent only if $\alpha=0$. Sen's model universe must therefore, in view of his (2.16), degenerate from the Einstein universe to the empty universe of special relativity, a conclusion which he overlooks. He proceeds instead to obtain a red-shift formula for $\alpha\neq 0$, by what appears to be a misuse of the term "the origin." In (1), which is Sen's (2.27), "the origin" means the point at which $S=0$; but in his red-shift formula (2.28) S becomes without proof "the metric interval between the observer (at the origin) and the particle," so that now "the origin" apparently means $r=0$. It is easy to show from Sen's equations that the world-line of an observer O at $r=0$ has no point in common with the world-line of P at $r=a$. Sen also proves from his equation (2.29) for the motion of a light-ray travelling from P to O that, if the period in t-time of the wave at emission is dt_1 and if the period at receipt by 0 is dt_0, then $dt_1=dt_0$. Thus if s, s' are the proper-times along P's and O's world-lines, respectively, then $ds/dt=1$, $ds'/dt=1$ and so $ds_0'=ds_1$. Hence no red-shift is predicted whether proper-time or t-time periods are employed. However Sen apparently believes that periods measured in terms of the S, S' parameters of P's and O's world-lines ought to be used. Since, for O as for P,

$$\frac{dS'}{dt} = (1-i\alpha S')^{-1} \tag{3}$$

the S-periods, dS_1 and dS_0', can be proved to satisfy, by Sen's(2.29) and (1), (2), (3) above,

$$\frac{dS_0'}{dS_1} = e^{-i\alpha(t_0-t_1)} = e^{-i\alpha(s_0-s_1)},$$

where $t_0-t_1=s_0-s_1$ is the time of travel of the light and $t=0$ corresponds to both $S=0$ and $S'=0$. This formula (which Sen omits to deduce) again shows that $\alpha=0$ because, however the periods of a light-wave at different moments in its history may be measured, these periods cannot have a ratio which is a complex number. And if $\alpha=0$, then again $dS_0'=dS_1$ and so there is no red-shift. This reviewer's conclusion is that, if Sen's theory is correctly worked out, no red-shift would be expected in his statical model universe.

G. C. McVittie.

Arsac, J. Application of mathematical theories of approximation to aerial smoothing in radio astronomy. Austral. J. Phys. **10** (1957), 16–28.

The gain function of a finite linear array of aerials of length L consists of a finite Fourier series of $n+1$ terms associated with a fundamental length a. If a is of the order

of many wavelengths λ, $n=L/a$. It follows that the image of a distribution of radio brightness will contain only those harmonics that characterize the array; the Fourier components of the distribution are weighted by the Fourier coefficients of the array and can be determined from an analysis of the image distribution. The consequences of these properties of an array are examined in this paper.

By Bernstein's theorem it is shown that the half-power points of the reception pattern cannot lie within an angular distance $\sqrt{2}\lambda/2\pi L$ of the direction of maximum gain, a closer limit than the value λ/L usually taken. The accuracy with which the image represents the object observed is considered. A lower limit is derived for the difference between image and object brightness at all points. For a given length L this is not necessarily reduced by decreasing a and thus increasing n. Suggestions are made as to designs which will reduce the Gibbs oscillations of the Fourier sum. Although the array rejects higher harmonics of the object distribution, it is possible to reconstitute an improved image by suitable weighting of the components that are accepted. Whilst systematic errors can be reduced by the design of an array of given length L, this has the effect of enhancing the errors due to inaccuracy of measurements, so that in practice a suitable compromise has to be found. *K. C. Westfold* (Pasadena, Calif.).

See also: Probability: Münch. **Optics, Electromagnetic Theory, Circuits:** Sturrock.

Geophysics

Sekerž-Zen'kovič, T. Ya. On the propagation of a tidal wave in a strait. Izv. Akad. Nauk SSSR. Ser. Geofiz. **1957**, 493–503. (Russian)

The author investigates tidal wave propagation in a strait, under the simple assumptions that definite cycles of oscillations of the level are given at both estuaries. The propagation of these oscillations is studied. The simplest time-periodic oscillations are analyzed where the channel length is considered to be arbitrary. The channel length is determined and the solution is carried through for simple enough calculations. Cotidal maps of the tidal wave propagation are constructed for a number of cases. *M. D. Friedman.*

Adachi, Ryuzo. A method of exploration on the seismic prospecting. I. When the curvature of time-distance curve is small concerning refraction method. Kumamoto J. Sci. Ser. A. **3** (1957), 1–19.

If the separation curve (vertical section of a discontinuity interface) of two layers can be approximated by a straight line, then the corresponding time-distance curve is, in the first approximation, also a straight line. In this paper the interpretation problem for this special case of a seismic profile is studied. *E. Kogbetliantz.*

Adachi, Ryuzo. A method of exploration on the seismic prospecting. II. General case concerning the refraction method. Kumamoto J. Sci. Ser. A. **3** (1957), 20–24.

By an elementary mathematical procedure the author deduces the parametric equations of the curve of separation (mirror under the shooting profile) of two layers. They contain the velocities v_1 and v_2 of seismic waves in two layers and an integration constant, the determination of which necessitates the knowledge of the depth to the mirror at one point. If this depth and the two velocities are known, the problem of interpretation of the seismic profile is theoretically solved. Unfortunately the parametric equations contain the derivative $F'(x)$ of the function $F(x)=t$ which represents the equation of the time-distance curve observed on the profile. Experimental values of $F(x)$ for some values of x only are known which makes a correct evaluation of its derivative practically impossible. *E. Kogbetliantz* (New York, N.Y.).

Adachi, Ryuzo. Fundamental relations on the seismic prospecting and a method of exploration. III. General case concerning the reflection method. Kumamoto J. Sci. Ser. A. **3** (1957), 25–31.

Let $t=F(X)$ denote the equation of an observed time-distance curve and let $y=y(x)$ represent the equation (unknown) of the corresponding velocity discontinuity curve of separation (mirror) of two layers. Elementary relations based on the hypothesis of a constant known velocity are solved for $y=y(x)$. The expression obtained in terms of $F(X)$ and X contains the derivative dF/dX. Small errors in dF/dX cause big errors in values of $y(x)$ and they cannot be avoided because experimental values of $F(X)$ are known for some X only. *E. Kogbetliantz.*

Bahvalov, S. V.; and Židkov, N. P. The direct geodesic problem. Vestnik Moskov. Univ. Ser. Mat. Meh. Astr. Fiz. Him. **12** (1957), no. 2, 15–23. (Russian)

A rapidly converging iteration method is presented to evaluate the arc length integral for a geodesic line on the terrestial sphere. Convergence and certain approximation procedures are examined in detail. *N. A. Hall.*

Starkl, L. Über vektographische vermittelnde Koordinatenausgleichung bei der Einzelpunkteinschaltung in gezwängte Triangulationsnetze. Österreich. Z. Vermessgwes. **45** (1957), 41–48, 81–87, 129–135.

Die Koordinaten eines trigonometrischen Punktes, der in einem vorhandenen Netz durch Richtung- oder Winkelmessungen eingeschaltet wird, sind in der Regel auf rechnerischem Wege nach der Methode der kleinsten Quadrate zu ermitteln. Hier dagegen wird eine graphische Lösung vorgeschlagen: Die aus der Minimumsforderung fließenden Normalgleichungen werden als Vektorsumme gedeutet und graphisch konstruiert, wobei eine Abbildung mit dem doppelten Richtungswinkel erforderlich wird. Nach Ausführung von 3 solcher Konstruktionen gelingt es, die unbekannten Koordinatendifferentiale — gegenüber einem willkürlichen Näherungspunkt — zu ermitteln. Ähnlich werden auch der Betrag des mittleren Richtungs-Fehlers sowie die Fehlerellipse bestimmt. Zusätzliche Schwierigkeiten müssen beim Rückwärtseinschneiden behoben werden, um die Orientierungsunbekannte als additive Konstante für die Werte des Richtungsbüschels zu berücksichtigen. *H. Wolf* (Bonn).

See also: Statistics: Aki.

OTHER APPLICATIONS

Economics, Management Science

Tobin, James. Estimation of relationships for limited dependent variables. Econometrica **26** (1958), 24–36.

This paper considers the statistical estimation of a "variable which has a lower or upper, limit and takes on the limiting value for a substantial number of respondents; for the remaining respondents, the variable takes on a wide range of values above, or below, the limit." (Example: expenditures on automobiles during any given year by a household.) The model presented is a hybrid of probit analysis and multiple regression. The maximum likelihood equations are given and their solution by Newton's method is discussed. Initial trial estimates are obtained through a linear approximation to the equations. The application of this estimation method to tests of hypothesis by the likelihood ratio method is indicated.

M. J. Beckmann (New Haven, Conn.).

Hahn, F. H. Gross substitutes and the dynamic stability of general equilibrium. Econometrica **26** (1958), 169–170.

It is proved that if all goods (including numeraire) are gross substitutes, then the usual adjustment process for general equilibrium systems is locally stable. This substantially extends some results of Metzler [Econometrica 13 (1945), 277–292; MR 7, 465]. *R. Solow.*

Arrow, Kenneth J. Statistics and economic policy. Econometrica **25** (1957), 523–531.

Presidential address to the Econometric Society.

R. Solow (Cambridge, Mass.).

Freeman, Raoul J. Ss inventory policy with variable delivery time. Management Sci. **3** (1957), 431–434.

An extension of T. M. Whitin's [The theory of inventory management, Princeton, 1953] constant delivery time inventory work is given. Here the author allows variable delivery and obtains a parallel criterion to minimize the total variable cost which is the sum of ordering cost, storage cost and depletion cost. The method yields a policy which is practically useful though not necessarily optimal in the class of all policies [see Dvoretzky, Kiefer, and Wolfowitz, Econometrica **20** (1952), 187–222; MR 13, 856]. *T. L. Saaty* (Washington, D.C.).

Johnson, S. M. Sequential production planning over time at minimum cost. Management Sci. **3** (1957), 435–437.

The paper is concerned with the following problem of production planning: Given the time-dependent costs of producing and storing a certain commodity, to schedule production over time so as to minimize the total costs while meeting a given schedule of requirements. The unit cost of production is supposed to be an increasing function of the production rate. It is shown that the total cost is minimized if each unit requirement is met sequentially in order of its due date by assigning the cheapest unit cost (producing and storing) available at that stage.

W. Prager (Providence, R.I.).

Koopman, B. O. The theory of search. II. Target detection. Operations Res. **4** (1956), 503–531.

[For part I see Operations Res. **4** (1956), 324–346; MR 18, 451.] Examples illustrating the calculation of geometric probabilities of visual or radar detection for various patterns of motion of searcher and target. *J. Kiefer.*

Koopman, Bernard O. The theory of search. III. The optimum distribution of searching effort. Operations Res. **5** (1957), 613–626.

If $p(x)$ is the probability density of location of a target and $\int_A \phi(x)dx$ is the length of time spent in searching any subset A of the line, then, subject to (*) $\int_{-\infty}^{\infty} \phi(x)dx=c$, the probability $\int_{-\infty}^{\infty} \exp[\log p(x)-\phi(x)]dx$ of not detecting the target is minimized by choosing $\phi(x)=[\log p(x)-b]^+$, where b is chosen so as to satisfy (*). *J. Kiefer.*

Bellman, Richard. Terminal control, time lags, and dynamic programming. Proc. Nat. Acad. Sci. U.S.A. **43** (1957), 927–930.

The author considers the "control process" problem of minimizing with respect to the n-vector function v the quantity $\int_0^T h(x(t))dG(t)+\int_0^T k(v(t))dt$ under restrictions $R_i(x, v)\leq m$, $1\leq i\leq m$, where h, k, and G are given and x satisfies $dx/dt=Ax+v$, $x(0)=c$. He outlines a computational technique which in many cases reduces the problem to consideration of a sequence of functions of lower dimension than n. *J. Kiefer* (Ithaca, N.Y.).

Goode, Harry H. The application of a high speed computer to the definition and solution of the vehicular traffic problem. Operations Res. **5** (1957), 775–793.

This paper gives a brief account of some simulation studies of the flow of highway traffic through intersections. Mostly, however, it deals with a vague and controversial comparison of various approaches to the study of highway traffic, leading to a conclusion by the author that detailed simulation by computers is the most promising approach. *G. Newell.*

Karush, William. A queuing model for an inventory problem. Operations Res. **5** (1957), 693–703.

In the problem considered, the number of items in stock, plus those being manufactured for replenishing stock, is a constant, n. Demand is considered to be a Poisson process with parameter λ, and the probability distribution of the replenishing period is known. The problem is equivalent to a queueing problem considered by Palm. A formula for $p(0|n)$, the steady-state probability of being out of stock at the instant of a demand, is obtained, and shown to be a convex decreasing function of n. Implications to relative allocation of inventory space to competing commodities are discussed. *E. Reich.*

Jaeckel, K. Verteilungen von Warte- und Anschlusszeiten. Z. Angew. Math. Mech. **37** (1957), 404–406.

Two processes of respective durations x, y, must be completed before a third process can begin. The random variables x, y, are independent and normal. The "waiting time" of the third process is $z=|u|$, $u=x-y$. Another variable of interest is $t=\max(x, y)$. The author considers the distributions of z and t, and related estimation problems. The work could have been greatly condensed if the author had realized that u was normal, and that z^2 and u^2 had the same distribution. *E. Reich.*

Arrow, Kenneth J. Decision theory and operations research. Operations Res. **5** (1957), 765–774.

The author discusses in general terms one of the funda-

mental areas of operations research, the study of multi-stage decision processes. As an application of the important concept of the interrelation of the various aspects of model-building, he discusses a particular dynamic programming process, the "optimal inventory" process. In conclusion, he discusses different ways in which the idea of successive approximations enters in the determination of a computational solution. *R. Bellman.*

Isbell, J. R. An optimal search pattern. Naval Res. Logist. Quart. **4** (1957), 357–359.

In Bull. Amer. Math. Soc. **62** (1956), 270, the reviewer posed some variational problems of unconventional type of which the following is typical: Given a base point p and a line L at unit distance from p in a plane, to find a rectifiable curve C of minimum length such that C joins p to L, and if C is rotated about p through any angle, it still meets L. The author solves this problem in an ingenious fashion and makes some brief comments about the corresponding problem when the point p is located between two parallel lines. *R. Bellman.*

Preinreich, Gabriel A. D. Replacement in the theory of the firm. Metroecon. **5** (1953), 68–86.

Let I be the capital value of a firm, to be maximized T_j dates, dividing the future into segments; $f=r(x, \dot{x}, \ddot{x}, \cdots)e^{-it}$ the discounted income stream (x cumulative output, i force of interest), W capital outlays (discontinuous),

$$I=\sum_{j=1}^{\infty}\Big[\int_{T_{j-1}}^{T_j} f_j dt - W_{j+1}e^{-iT_j}\Big].$$

The terminal value of any segment must be the initial value of the next. This theory is illustrated by examples.

With complete replacement, let r_1 be the income in the first period, G_j goodwill at the beginning of the jth time period. Now the maximand is:

$$I=\int_0^{T_2}(r_1-i\sum_{j=2}^{\infty}G_j)e^{-it}dt+\sum_{j=2}^{\infty}G_j.$$

Welfare aspects of the theory are also discussed with the help of the concept of consumer's surplus. A parameter λ measures the discount or premium placed on consumer's surplus as compared to the value of cash. *G. Tintner* (Ames, Iowa).

Clark, Charles E. Mathematical analysis of an inventory case. Operations Res. **5** (1957), 627–643.

See also: Probability: Cramér; Barrer. **Statistics:** Briggs. **Programming, Resource Allocation, Games:** Bellman; Bellman, Clark, Malcolm, Craft and Ricciardi. **Information and Communication Theory:** Brotman and Minker.

Programming, Resource Allocation, Games

★ **Bellman, Richard. Dynamic programming.** Princeton University Press, Princeton, N. J., 1957. xxv+342 pp. $6.75.

"Dynamic Programming" is the term coined by the author for the study of multi-stage decision processes and, in particular, for the functional equation approach to finding optimum policies. This kind of approach, as the author points out, is intimately related to the invariance principle found in certain developments in physics (e.g., light scattering); more recently, the characterization of Bayes strategies in sequential statistical decision problems by Wald and Wolfowitz (and the related optimum character of the sequential probability ratio test in statistics) was another predecessor of dynamic programming which used this approach. The approach is merely to note that an optimal policy must have the property that, whatever the initial state of the system and initial decision, relative to the resulting state (new initial state) the remaining decisions must constitute an optimal policy. This principle makes it simple to write down a functional equation for the total gain to be achieved from using an optimal policy (as a function of the initial state) for N stages, in terms of the corresponding function for $N-1$ stages and appropriate transformations and costs which describe the transition from one stage to the next; in the stationary case of infinitely many periods, the functional equation relates the function to itself. Every functional equation in the book is obtained easily in this way. The real problem starts after the equation is written down: to solve it and describe the associated optimal policy. The author has been a leading developer of techniques for solving such functional equations. The book is a collection of methods, uniqueness and existence theorems, and examples, which have for the most part appeared in the literature in recent years and are due to various people, but most often to the author. The appropriate references are too many to cite in the description which follows.

Chapter 1, "A Multistiage Allocation Process", begins with the simple problem of dividing a quantity x into two nonnegative parts y and $x-y$, for which one will receive a return $g(y)+h(x-y)$ (g and h continuous) and, as a consequence of the division, will have at his disposal an amount $ay+b(x-y)$ $(0\leq a, b<1)$ to divide at the next stage in the same way. We are led to the functional equation

$$f(x)=\max_{0\leq y\leq x}[g(y)+h(x-y)+f(ay+b(x-y))]$$

for the maximum attainable over infinitely many stages. Existence and uniqueness of the solution are proved, and the approximations of f and of the optimum policy by the corresponding functions for the N-stage problem, as $N\to\infty$, are given. For the case when g and h are convex or concave, it is proved by induction on N and passage to the limit (one of the standard techniques of the book) that f will be of the same form; this yields information on the form of the optimal policy. Particular examples are given. A stability theorem (how much does f vary with g and h?) is proved. Time dependence, multidimensional generalizations, and other comments and examples conclude the chapter.

Chapter 2, "A Stochastic Multistage Decision Process", considers the problem of determining the best mining policy when there are two mines, the ith having an amount x_i of gold in it, and where at each stage if we choose mine i we will receive a fraction r_i of the gold in that mine with probability p_i and will damage our equipment beyond repair with probability $1-p_i$. The format is that of Chapter 1, the optimum strategy at each stage being to choose i to maximize $r_ip_ix_i/(1-p_i)$. Generalizations and the N-stage problem are treated.

Chapter 3, "The Structure of Dynamic Programming Processes", discusses and generalizes the simple examples of the previous chapters and sets forth the general approach.

Chapter 4, "Existence and Uniqueness Theorems", makes use of the shrinking nature of the transformation appearing in various functional equations to prove results of the type proved previously in the special case of Chapter 1.

Chapter 5, "The Optimal Inventory Equation", proves that the optimal inventory policy is of the form "order up to S if less than S is on hand, otherwise order nothing", provided ordering and penalty costs are linear and homogeneous. Multidimensional generalizations and the N-stage problem are treated. Other cost functions are treated; in particular, the cases of time lag and of convex ordering cost are treated by once more using induction on N and passing to the limit. An appendix on the renewal equation is included.

Chapter 6, "Bottleneck Problems in Multistage Production Processes", treats problems where production is limited at each stage by capacity and availability of goods from previous production stages. The discrete case (N time periods) leads to a linear programming problem. The continuous time case (up to time T) leads to the problem of maximizing the inner product of vectors a and $x(T)$ subject to $dx/dt=Ax+Bz$, $x(0)=c$, and (for $0\leq t\leq T$) $z(t)\geq 0$, $Cz(t)\leq Dx(t)$, where A, B, C, D are matrices. A simple example is treated.

Chapter 7, "Bottleneck Problems Examples", applies the discussion of the previous chapter to an example of automobile and steel production.

Chapter 8, "A Continuous Stochastic Decision Process", treats the analogue of the example of Chapter 2 for continuous time. Here for the first time it is necessary to introduce randomized decision procedures. Differential equations describing the change of the system with time are considered, and the optimum procedure is obtained. The case where one wants to maximize a (utility) function of the gain is also considered. The case where a third choice — partial use of each mine — is also allowed yields to analysis, although it did not in the case of Chapter 2.

Chapter 9, "A New Formalism in the Calculus of Variations", is not as formidable a concept as its title suggests. The approach is merely that described at the outset of this review; thus, e.g., in the classical brachistochrone problem, it says that the minimum time path must have the property that any subarc is a minimum time path between its ends. The author is interested in minimizing or maximizing generalizations of the functional that appeared in Chapter 6 — functionals of x and z, these vectors being connected by a differential equation and being subject to certain constraints. The minimization problem associated with the eigenvalue problem $u''(t)+\lambda^2\varphi(t)u(t)=0$, $u(0)=u(1)=0$, is also considered. Discrete approximations to solutions are treated.

Chapter 10, "Multistage Games", is concerned with games where the resources of each of two players at a given stage depend on what has transpired before. Examples are games of pursuit and of survival. Existence and uniqueness theorems and approximations to solutions are again obtained. The author feels that in a non-zero-sum game of survial, there is now a satisfactory definition of optimum strategy — it is that strategy which maximizes the probability that a player will outlast his opponent [this point of view really changes the payoff function].

Chapter 11, "Markovian Decision Processes", considers briefly both the discrete and continuous time cases of processes for which a transition probability function governs the change of state with time; the continuous time case leads to a nonlinear differential equation.

An important part of the book is the collection of 370 "Exercises and Research Problems" which follow the chapters. Although these examples are sometimes artifical or dull generalizations, the many good ones among them constitute the most interesting part of the book, in the reviewer's opinion. The text only considers a few examples in great detail, and these further exercises cover a wide span of applications of the author's techniques. The author apparently did not think it mattered to distinguish "Exercises" from the "Research Problems" in any way.

The book is intended as a text suitable at the advanced calculus level in a mathematic department or elsewhere. (A second volume on a higher mathematical level is planned.) As a text, the book enjoys (especially in the early chapters) the benefit of simple examples and lucid discussions of what is going on. It seemed to this reviewer that the repeated format of similar existence and uniqueness theorems and approximations in chapter after chapter would be unsuitable in a course for people interested in applications. Mathematically, there is not, at least in this volume, much of a deep conceptual nature; the book will definitely be of greatest benefit to the engineer or economist who wants an idea of how to attack various applied problems.

One regretable omission is that of the case where the chance laws are unknown. Such problems are usually extremely difficult to treat, but they are certainly very important from a practical point of view. *J. Kiefer.*

Karush, W.; and Vazsonyi, A. Mathematical programming and employment scheduling. Naval. Res. Logist. Quart. **4** (1957), 297–320.

The authors consider the problem of planning employment levels w_i for time periods $i=1, \cdots, N$ in order to meet specified production requirements r_i. There are costs of the various levels w_i and of changing the level by an amount w_i-w_{i-1} between periods, and the total cost over all periods is to be minimized. Under certain assumptions on the cost function, the latter is broken up into pieces, the consideration of suitable envelopes of which leads to an equivalent problem for which a computing algorithm is given. The details are hard to summarize briefly. *J. Kiefer* (Ithaca, N.Y.).

Rubinstein, G. Sh. A generalization of the problem concerning the extreme intersection point of an axis with a convex polyhedron. Dokl. Akad. Nauk SSSR (N.S.) **113** (1957), 987–990. (Russian)

In a previous paper [G. Sh. Rubinstein, same Dokl. (N.S.) **100** (1955), 627–630; MR **17**, 133], the author has studied the problem of finding the extreme intersection point of a straight line with a convex polyhedron. Guided by practical consideration (e.g. advantage in less laborious computations), the present paper discusses a slightly more general formulation of the problem. The main part of the work is devoted to two particular linear programming problems, one of which is the Hitchcock transportation problem [F. L. Hitchcock, J. Math. Phys. **20** (1941), 224–230; MR **3**, 11; M. M. Flood, Pacific J. Math. **3** (1953), 369–386; MR **15**, 473]. The reduction of these concrete problems to the geometric extremization problem is explained with some detail. *Ky Fan* (Oak Ridge, Tenn.).

★ **Motzkin, T. S. The assignment problem.** Proceedings of Symposia in Applied Mathematics. Vol. VI. Numerical analysis, pp. 109–125. Published by McGraw-Hill Book Company, Inc., New York, 1956 for the American Mathematical Society, Providence, R. I. $9.75.

The problem of finding a permutation $p=\hat{p}$ such that the sum

$$a_{\hat{p}}=\operatorname{Max}_{p} a_p=\sum_i a_{ip_i}$$

is called the assigment problem. The material covered by Motzkin is presented under the subheadings: Normalization; Approximate solution; Vertex-approach and face-approach methods; The equidistribution problem; Related problems. A summary of geometric concepts used and a large bibliography is included. The subheading Normalization considers the question about the set of matrices $\mathfrak{d}$ with the property $d_{\hat{p}}=\operatorname{Max} d_p$, $a_{\hat{p}}=\operatorname{Max}_p a_p$. Further, he makes the remark that $e^{a_{\hat{p}}}$ gives the absolutely greatest term in the expansion of the determinant of $\{e^{a_{ij}}\}$. As approximate solutions he considers permutations p for which a_p is approximately equal to $a_{\hat{p}}$. He presents methods for finding upper and lower bounds for $a_{\hat{p}}$. As a good lower bound he mentions $t=\sum_{k=1}^{n} a_{i_k j_k}$, where $a_{i_k j_k}=\operatorname{Max}_{i,j} a_{ij}$; $i \notin I_{k-1}$; $j \notin J_{k-1}$; $i \in I_{k-1}$ if $i=i_\nu$; $\nu<k$ and $j \in J_{k-1}$ if $j=j_\nu$, $\nu<k$. Further, he tries to make use of the fact that for $p=\hat{p}$ we have $a_{i\hat{p}_i}+a_{i'\hat{p}_{i'}} \geqq a_{i\hat{p}_{i'}}+a_{i'\hat{p}_i}$, but he does not hit upon the idea of defining the lower limit for $a_{\hat{p}}$ by the very efficient formula

$$a_{\hat{p}} \geqq \sum_\nu a_{i_\nu j_\nu}=a_{p'}$$

where $a_{i_\nu j_\nu}+a_{h_\nu k_\nu}=\operatorname{Max}(a_{ij}+a_{hk})$ for $a_{ij} \geqq a_{hk}$, $i \neq h$, $j \neq k$, $i \in I_{\nu-1}$, $h \in I_{\nu-1}$, $j \notin J_{\nu-1}$, $k \notin J_{\nu-1}$. Very often we will then have $a_{\hat{p}}=a_{p'}$. If this is not the case it is often easy to correct the permutation p' to a permutation p'' such that the relations $a_{ip''_i}+a_{i'p''_{i'}} \geqq a_{ip''_{i'}}+a_{i'p''_i}$ will be fulfilled for all pairs (i, i') in which case we almost always will have $a_{\hat{p}}=a_{p''}$. The other parts of Motzkin's paper consider questions which are somehow related to the assignment problem without being directly useful for solving such problems in practice. *L. Törnqvist.*

★ **Förstner, Karl; und Henn, Rudolf. Dynamische Produktions-Theorie und Lineare Programmierung.** Mit einem Vorwort von Prof. Dr. W. Waffenschmidt. Schriften zur wirtschaftswissenschaftlichen Forschung, Bd. 5. Verlag Anton Hain, Meisenheim-Glan, 1957. 125 pp. DM 14.50.

This is an elementary account of static and dynamic production (simple and joint). Quality variation of the product is considered. Static and dynamic problems of linear programming are solved by the simplex method. *G. Tintner* (Ames, Iowa).

Braumann, Pedro. Theorem about systems of linear equations. Univ. Lisboa. Revista Fac. Ci. A. (2) **5** (1955–1956), 103–112 (1956)=Inst. Alta Cultura. Estudos e Notas Sem. Mat. Fac. Ci. Lisboa Publ. no **22** (1955), 10 pp.

Sufficient conditions are formulated on an n by $n+1$ matrix (a_{ij}) for the uniqueness and positivity of the solution of $\sum_{j=1}^{i} a_{ij}z_j-\sum_{j=i+1}^{n} a_{ij}z_j=a_{i,n+1}$. They involve all a_{ij} and certain 2 by 2 subdeterminants being non-negative with the main diagonal strictly positive. The hypotheses are abstracted from a more special problem which is said to arise in multi-stage decision theory. *J. Isbell* (Seattle, Wash.).

Charnes, A.; and Cooper, W. W. Nonlinear power of adjacent extreme point methods in linear programming. Econometrica **25** (1957), 132–153.

The simplex method and its variants consist of movement from an extreme point to an adjacent extreme point of the solution-set; an optimum, if it exists, must be obtainable at an extreme point, and all optima can be generated from the extreme point optima. These methods are therefore called "adjacent extreme point methods". They have previously been shown to be applicable to problems in which certain restricted classes of convex non-linear functionals were to be minimized subject to linear constraints. It is the aim of this paper: (1) to show that the class of functionals can be extended, including functionals which need not be either convex or concave; (2) to characterise the class of functionals to which adjacent extreme point methods can be applied, (3) to indicate modifications in the simplex procedures and criteria of optimality which are necessary to effect this extension; (4) to assess the limit of such extensions and, finally, (5) to suggest additional research.

A detailed example is worked out in the appendix. *W. F. Freiberger* (Providence, R.I.).

Bartlett, T. E. An algorithm for the minimum number of transport units to maintain a fixed schedule. Naval Res. Logist. Quart. **4** (1957), 139–149.

Thompson, Gerald L. Decision making and new mathematics. Naval Res. Logist. Quart. **3** (1956), 141–149 (1957).

A lecture, addressed to a conference on methods in philosophy and the sciences, in which applications of the theory of games are discussed with a minimum of formal mathematics. *W. F. Freiberger* (Providence, R.I.).

Grenander, Ulf. A prediction problem in game theory. Ark. Mat. **3** (1957), 371–379.

To predict $\int_0^1 a(t)x_t dt$, a a known continuous function, $\{x_t\}$ a stationary process of mean 0 and variance 1, observed for $t \leqq 0$; so as to minimize maximum expected square of error. It is shown that min max=max min= largest eigenvalue of the kernel $K(x, y)$ defined as follows. Let $m=\min(1-x, 1-y)$. Then

$$K(x, y)=\int_0^m a(x+u)a(y+u)du.$$

The cases $a(t) \equiv 1$ and $a(t)=1-t$ are worked out. The heart of the theorem is the reduction to an inequality, proved in Grenander, Ulf; and Szegö, Gabor, Toeplitz forms and their applications. University of California Press, 1958. *J. Isbell* (Seattle, Wash.).

Nolfi, P. Hinweise auf die Ergebnisse und Bedeutung der Spieltheorie. Mitt. Verein. Schweiz. Versich.-Math. **57** (1957), 129–144.

This is an expository paper on the theory of games of strategy. The definitions of strategy and normal form of a game are given. The fundamental theorem of two-person, zero-sum, finite games is stated and illustrated. The paper contains a brief survey of n-person games and games against Nature. *M. Dresher.*

Bellman, Richard; Clark, Charles E.; Malcolm, Donald G.; Craft, Clifford, J.; and Ricciardi, Franc M. On the construction of a multi-stage, multi-person business game. Operations Res. 5 (1957), 469–503.

Hale, J. K.; and Wicke, H. H. An application of game theory to special weapons evaluation. Naval Res. Logist. Quart. 4 (1957), 347–356.

See also: Numerical Methods: Ward; Bellman. **Information and Communication Theory:** Bellman and Kalaba.

Biology and Sociology

Rapoport, Anatol. Contribution to the theory of random and biased nets. Bull. Math. Biophys. 19 (1957), 257–277.

The general problem is to determine the frequency of occurrence of closed cycles, the distribution of chain lengths, and the like, given certain assumptions as to the connections between pairs. Assumptions appropriate to various situations are considered, some approximate formulas are derived and related to an empirically determined sociogram. *A. S. Householder.*

Shepard, Roger N. Stimulus and response generalization: A stochastic model relating generalization to distance in psychological space. Psychometrika 22 (1957), 325–345.

The author considers a learning situation in which a subject is presented with stimuli S_i taken in random order from some set, and is permitted responses R_j from some specified set. For each subject, however, a one-to-one correspondence is assigned by the experimenter between stimulus and response, the particular response R_j corresponding to S_i being the "correct" one for that stimulus for that subject. It is assumed that errors result from either a confusion (i.e. generalization) of stimuli by the subject, a confusion (i.e. generalization) of responses, or both.

Thus at any stage of the learning process for a given subject, the probability of occurrence of a particular R_j following a particular S_i is expressed in terms of stimulus generalization and response generalization. Each of these, in turn, is expressed as a function of time, representing the stage of the learning process, and of psychological distances between stimuli and between responses, respectively. An exponential relationship is assumed between distances and probabilities, and problems of estimation are discussed. *A. S. Householder.*

Simon, Herbert A. Amounts of fixation and discovery in maze learning behavior. Psychometrika 22 (1957), 261–268.

A model is constructed to fit certain learning situations, for example maze learning. A maze is characterized by n, the number of alternate paths at each choice point and L, the number of choice points. Measures of learning are E, the total number of errors to criterion and T the number of trials to criterion. Under certain simple assumptions, suggested from experimental results, it is proved that $E=a(n-1)LT$ for some constant a. Experimental results further suggest that T is proportional to L, giving $E=a'(n-1)L^2$. Data is shown to fit this model.

J. L. Snell (Hanover, N.H.).

Cotton, John W.; Campbell, Donald T.; and Malone, R. Daniel. The relationship between factorial composition of test items and measures of test reliability. Psychometrika 22 (1957), 347–357.

The authors first indicate that there is a close relationship between the statistical psychological concepts of reliability and factor structure as pointed out by Cronbach in his examination of the four different kinds of reliability coefficients. They present a theory, for the continuous case, for a general coefficient of equivalence, similar to that of Cronbach, which is a generalization of the Kuder-Richardson Coefficient of equivalence. In contrast to Cronbach, they argue that dichotomous items can not have the factor structure implied by the model for the continuous case, so that direct interpretation of the coefficient of equivalence is impossible. However, for continuous distributions associated with dichotomous item scores, the proportion of common-factor variance may be expressed as a function of the intercorrelations among the items. An empirical study with test data from populations of different factorial structure leads them to the observation that "Within the limits of the samples employed, our estimates are quite satisfactory".

P. S. Dwyer (Ann Arbor, Mich.).

Information and Communication Theory

Kelly, J. L., Jr. A new interpretation of information rate. Bell. System Tech. J. 35 (1956), 917–926.

The author considers the fate of a gambler who bets on an event with binary outcome and even money pay-off. If V_N is the gambler's capital after N bets, then $G=\lim_{N\to\infty}(1/N)\log V_N/V_0$ is called the exponential rate of growth of his capital, and if he knows the outcome of the binary event in advance, he can bet to make $V_N=V_0 2^N$ and $G=1$. If he receives information about the outcome over a noisy channel which gives correct results with probability q and if he bets a fixed fraction b of his capital each time, then $b=2q-1$ maximizes G with probability one and $G\max=1+q\log q+(1-q)\log(1-q)=1-H=R$ =rate of transmission as defined by Shannon [same J. 27 (1948), 379–423, 623–656; MR 10, 133]. Non-binary cases are considered and similar expressions involving Shannon's rate of transmission are obtained.

R. A. Leibler (Silver Spring, Md.).

★ **Черенин, В. П. [Čerenin, V. P.] Некоторые проблемы документации и механизация информационных поисков [Some problems of documentation and mechanization of information searches.]** Inst. Nauč. Informacii Akad. Nauk SSSR, Moscow, 1955. 76 pp. 4 rubles.

There are 10 sections, on the topics: preservation and distribution of information (with a historical sketch), search for information by means of indexes, deficiencies of indexes, mechanization of information searches; information language and indexing, coding, devices with parallel action (with several illustrations of punched and notched cards, etc.), devices with sequence action (with illustrations of flexowriters, etc.), devices with sequence action with continuous carrier (with illustrations of magnetic drum, etc.).

Middleton, David; and Van Meter, David. Detection and extraction of signals in noise from the point of view of statistical decision theory. II. J. Soc. Indust. Appl. Math. **4** (1956), 86–119.

This is the continuation of part I [same J. **3** (1955), 192–253; MR **18**, 180]. Smoothing, estimation, and prediction are discussed from the point of view of decision theory. Specific computations for such cases as "determinate signals" and gaussian noise illustrate the general situations.

In section 5, detectors that minimize equivocation, in the sense of Shannon, are considered. The resulting optimum detectors are then compared to detectors designed on the bases of more classical criteria. *E. Reich.*

Brotman, Lewis; and Minker, Jack. Digital simulation of complex traffic problems in communications systems. Operations Res. **5** (1957), 670–679.

This is a report on the use of a digital computer for simulating the operation of a telephone communications system. The effort is motivated by two circumstances: (i) the lack of amenable mathematical models which are global in character, that is, do not assume that the system consists of small, independent subsystems; and (ii) a desire to explore the digital computer as a flexible means for evaluating choices of routes for calls, operator procedures, and the effect of breakdowns and other transients. Considering the complexity of the topic, the report is very brief, and the comparison of simulation results to analytic work is obscure. *V. E. Beneš.*

Bellman, Richard; and Kalaba, Robert. Dynamic programming and statistical communication theory. Proc. Nat. Acad. Sci. U.S.A. **43** (1957), 749–751.

A noisy channel transmits a sequence of independent signals chosen from amongst a finite number. The probability distribution of signals and the probabilities of wrong transmission are known to the receiver who, with given total capital, has to place bets on which signal was transmitted. Given the pay-off and the utility the authors state iterative functional equations for the optimum expected utility for betting on N stages. For the special case where the utility is the logarithm of the capital, the solution of the equations is given. Generalizations, in particular to correlated signals, are indicated.

D. V. Lindley (Cambridge, Mass.).

★ **Wozencraft, John McReynolds. Sequential decoding for reliable communication.** Research laboratory of electronics, Massachusetts Institute of Technology, Cambridge, Mass., Tech. rep. 325, August 9, 1957. vii+153 pp.

This doctor's thesis discusses error-correcting codes which are capable of transmitting at rates near the maximum for a binary symmetric channel. The coding and decoding processes for such codes are studied carefully, especially the average number of decoding computations necessary to operate such a signalling system.

The report contains a discussion of much of the relevant literature, and four appendices provide proofs of some of the more involved parts of the analysis.

R. Hamming (Murray Hill, N.J.).

See also: Measure, Integration: Wernikoff.

Control Systems

Letov, A. M. Conditionally stable control systems (concerning a certain type of optimum control systems). Avtomat. i Telemeh. **18** (1957), 601–614. (Russian. English summary)

The control system being discussed is described by a system

$$\dot{x}_k = \sum_{i=1}^{n} b_{ki}x_i + c_k\xi \quad (k=1, \cdots, n),$$
$$\dot{\xi} = Kx_1 \qquad (t \leqq t_*),$$
$$\dot{\xi} = -Kx_1 \qquad (t > t_*).$$

The sign of the speed $\dot{\xi}$ of the servomotor is switched at time t^*. The linear system corresponding to $t \leqq t_*$ is assumed to be asymptotically stable (the characteristic roots have negative real parts). After switching, the resulting system is assumed to have only one root with positive real part. Let $x(x_0, t)$ be the state of the system at time t. Assuming no switching takes place, let t^* be the time of conditional damping (i.e., $|x(x_0, t)| \leqq x_0 e^{-\pi}$ for $t \geqq t^*$). There then exists a switching time t_* ($0 < t_* < t^*$) such that the new system is conditionally stable, and it appears from the examples given that the time of conditional damping is greatly reduced. A method of control is then discussed in which a special computer — whose structure is discussed — determines the switching time t_*.

J. P. LaSalle (Notre Dame, Ind.).

Popov, E. P.; and Hlypalo, E. I. Evaluation of quality and choice of parameters of non-linear automatic control systems of high order. Izv. Akad. Nauk SSSR. Otd. Tehn. Nauk **1956**, no. 12, 30–47. (Russian)

A nontinuation of earlier work on damped oscillations in nonlinear control systems [Popov, same Izv. **1956**, no. 9, 3–23; MR **18**, 549]. *L. A. Zadeh* (New York, N.Y.).

★ **Svoboda, Antonin. Graphico-mechanical aids for the synthesis of relay circuits. Aktuelle Probleme der Rechentechnik.** Bericht über das Internationale Mathematiker-Kolloquium, Dresden, 22. bis 27. November 1955, pp. 43–50. VEB Deutscher Verlag der Wissenschaften, Berlin, 1957.

Doll, H. G.; and Stout, T. M. Design and analog-computer analysis of an optimum third-order nonlinear servomechanism. Trans. A.S.M.E. **79** (1957), 513–523, discussion 523–525.

Servomechanisms with a governing differential equation $J\ddot{c}=T$ are studied, where $T=2T_m F(c, \dot{c})/\tau$ for $|T|<T_m$ and $T=T_m F(c, \dot{c})$ otherwise. The quantities J, T_m and τ are positive constants. The function $F(c, \dot{c})$ appears in the special form $F=\mathrm{sgn}[2T_m(c/J)+\dot{c}|\dot{c}|]$ of which the author claims that it provides an optimum response to transients. The paper describes an electro-optical function generator for F and discusses some interesting examples and laboratory investigations.

H. Bückner (Schenectady, N.Y.).

Gendreu, R. Les servomécanismes dans les calculateurs analogiques. II. Ann. Radioélec. **12** (1957), 363–381.

See also: Ordinary Differential Equations: Repin.

HISTORY, BIOGRAPHY

★ **Neugebauer, O. The exact sciences in antiquity. 2nd ed.** Brown University Press, Providence, R. I., 1957. xvi +240 pp. (14 plates). $6.00.

The first edition was reviewed in MR 13, 809. In the present edition many additions referring to recently obtained results have been made and there are two new appendices, one on Greek mathematics, the other on the Ptolemaic system and its Copernican modification.

Busulini, Bruno. La reciprocità alle sorgenti della analisi infinitesimale. Ann. Univ. Ferrara. Sez. VII. (N.S.) **5** (1955–1956), 59–67 (1957).

A discussion of continuity and discontinuity in ancient Greek mathematics.

Busulini, Bruno. Il ruolo dell'analogia nella teoria dei numeri reali secondo Dedekind. Ann. Univ. Ferrara. Sez. VII. (N.S.) **5** (1955–1956), 79–83 (1957).

A sequel to the paper reviewed above.

★ **Г. Н. Берман. [Berman, G. N.] Счёт и число (как люди учились считать). [Counting and number. (How man learned to count).] 6th ed.** Gosudarstv. Izdat. Tehn.-Teor. Lit., Moscow, 1956. 32 pp. 0.45 rubles.

A popular essay in six chapters: counting in remote antiquity, how letters were used to express numbers, where and how the present positional system arose, our number system with its advantages and disadvantages, how "astronomical" figures can be expressed in concise form, how to express "dwarf" numbers.

★ **Dilgan, Hâmit. Nassireddin Toussi grande scienziato matematico.** Atti dell'VIII Congresso Internazionale di Storia delle Scienze, Firenze, 3–9 Settembre 1956, pp. 183–191. Tipografia Giuseppe Bruschi, Firenze, 1958. 11 pp.

In recent times, emphasis has often been laid on the fact that the libraries of Istanbul contain an enormous mass of material, as yet never carefully examined, which is indispensable for a proper study of medieval mathematics and astronomy. The present article is a short biography of one of those authors about whom much more would probably be learned from a systematic study of this material. There is a discussion of his study of Euclid's fifth postulate, of his construction of astronomical tables, and of his numerous translations. A genealogical table, a bibliography and a constructed portrait are given.

S. H. Gould (Providence, R.I.).

Di Pasquale, Luigi. Le equazioni di terzo grado nei "Quesiti et inventioni diverse" di Nicolò Tartaglia. Period. Mat. (4) **35** (1957), 79–93.

★ **Struik, Dirk J. The origins of American science (New England).** Cameron Associates, New York, 1957. xiv+430 pp. $6.00.

A study of the growth of science, and of the corresponding sociological changes, in New England between the Revolution and the Civil War. Mathematics is discussed chiefly in connection with navigation and astronomy.

Anonymous. Geometrie non-euclidee. Civiltà delle Macchine **6** (1958), 67–78.

A brief historical sketch, followed by extracts from Bonola, La geometria non-euclidea (Zanichelli, Bologna, 1906) and Lobačevskiĭ, Pangeometria, 1855 and by the complete text of Riemann, Sulle ipotesi che stanno a fondamento della geometria, 1854.

Zubov, V. P. V. V. Bobynin and his works on the history of mathematics. Trudy Inst. Istor. Estest. Tehn. **15** (1956), 277–322. (Russian)

Belozerov, S. E. On some questions in the history of the theory of analytic functions. Trudy Inst. Istor. Estest. Tehn. **15** (1956), 169–205. (Russian)

Gagaev, B. M. Works of Kazan mathematicians on orthogonal systems. Uspehi Mat. Nauk (N.S.) **12** (1957), no. 4(76), 251–262. (Russian)

A discussion of 27 articles by 9 mathematicians, beginning with Lobačevskiĭ.

Vavilov, S. I. In memory of A. N. Krylov. Trudy Inst. Istor. Estest. Tehn. **15** (1956), 4–5 (1 plate). (Russian)

Ioffe, A. F. A. N. Krylov in the academy of sciences. Trudy Inst. Istor. Estest. Tehn. **15** (1956), 6–12. (Russian)

Smirnov, V. I. Mathematical works of A. N. Krylov. Trudy Inst. Istor. Estest. Tehn. **15** (1956), 13–23. (Russian)

Idel'son, N. I. Works of A. N. Krylov in astronomy. Trudy Inst. Istor. Estest. Tehn. **15** (1956), 24–31. (Russian)

Kravec, T. P. In memory of A. N. Krylov. Trudy Inst. Istor. Estest. Tehn. **15** (1956), 32–39. (Russian)

Šatelen, M. A. In recollection of conversations with Alekseĭ Nikolaevič Krylov. Trudy Inst. Istor. Estest. Tehn. **15** (1956), 40–45. (Russian)

Kol'cov, A. V. Organizatorial activity of A. N. Krylov in the academy of sciences in the years 1918-1920. Trudy Inst. Istor. Estest. Tehn. **15** (1956), 46–53. (Russian)

Krasotkina, T. A. From the correspondence of A. N. Krylov with S. O. Makarov, I. P. de Kolong, N. E. Žukovskiĭ and others. Trudy Inst. Istor. Estest. Tehn. **15** (1956), 54–168. (Russian)

These 8 articles, composed at various times, are collected here on the occasion of the 10th anniversary of the death of A. N. Krylov.

Kučmar, M. I. Nikolaĭ Nikolaevič Nazarov (on the tenth anniversary of his death). Uspehi Mat. Nauk (N.S.) **12** (1957), no. 4(76), 125–134 (1 plate). (Russian)

A biographical sketch, chiefly scientific, with 1 photograph and a bibliography of 35 entries.

Ahiezer, N. I.; and Efimov, N. V. Boris Yakovlevič Levin (on the fiftieth anniversary of his birth). Uspehi Mat. Nauk (N.S.) **12** (1957), no. 2(74), 237–242 (1 plate). (Russian)

A biographical sketch, chiefly scientific, with 1 photograph and a bibliography of 41 entries.

Lavrent'ev, M. A.; and Sobolev, S. L. Il'ya Nestorovič Vekua (on the fiftieth anniversary of his birth). Uspehi Mat. Nauk (N.S.) **12** (1957), no. 4(76), 227–234 (1 plate). (Russian)

A bibliographical sketch, chiefly scientific, with 1 photograph and a bibliography of 63 entries.

Sobolev, S. L. On the works of A. M. Lyapunov on potential theory. Prikl. Mat. Meh. **21** (1957), 306–308. (Russian)

A lecture delivered at a joint session of the Presidium of the Academy of Sciences, the divisions of technical and physical sciences of the Academy of Sciences, the Moscow University, the Moscow Mathematical Society, the Institute of Mechanics of the Academy of Sciences, and the Institute of Automatics and Telemechanics of the Academy of Sciences held on June 6, 1957 in Moscow in honor of the centenary of the birth of A. M. Lyapunov.

★ **Лаппо-Данилевский, И. А. [Lappo-Danilevskiĭ, I. A.] Применение функций от матриц к теории линейных систем обыкновенных дифференциальных уравнений. [Application of matrix functions to the theory of linear systems of ordinary differential equations.] Edited by V. I. Smirnov with an introductory essay.** Gosudarstv. Izdat. Tehn.-Teor. Lit., Moscow, 1957. 456 pp. (1 plate) 18.75 rubles.

A translation into Russian by I. P. Mysovskiĭ from the French of the complete collected works of Lappo-Danilevskiĭ published in the years 1934–36 in the Trudy Fiziko-Matematičeskogo Instituta imeni V. A. Steklova under the editorship of N. E. Kočin and V. I. Smirnov. In the present edition two articles have been omitted as being not immediately connected with the author's life-work on functions satisfying systems of linear differential equations. As it stands, the book contains 11 articles and 12 shorter notes. The author's defence of his dissertation in 1929 is also given.

Truesdell, C. The new Bernoulli edition. Isis **49** (1958), 54–62.

A description of the planned edition of 20–25 volumes of about 700 printed pages each. The first volume has already appeared [MR **16**, 781]. The author of the present article regards the undertaking of this Bernoulli edition as "the most important event in the history of mathematics in a quarter of a century".

Leonard Euler (on the 250th anniversary of his birth). Prikl. Mat. Meh. **21** (1957), 153–156 (2 plates). (Russian)

A scientific biographical sketch with one photograph.

Yuškevič, A. P. Life and mathematical achievement of Leonard Euler. Uspehi Mat. Nauk (N.S.) **12** (1957), no. 4(76), 3–28 (1 plate). (Russian)

A detailed general and scientific biographical sketch, with one photograph.

Gel'fond, A. O. On some characteristic features of the ideas of L. Euler in the field of mathematical analysis and his "Introductio in analysin infinitorum". Uspehi Mat. Nauk (N.S.) **12** (1957), no. 4(76), 29–39. (Russian)

In Euler's treatment of infinitesimals, two characteristics stand out clearly; the algebraic character of most of his work on analysis, and the wide use made of extremal principles. The present essay illustrates these characteristics, especially the former one, by Euler's various algorithms; it also discusses Euler's ideas about infinity.

★ **Eulerus, Leonhardus. Opera omnia. Series secunda. Opera mechanica et astronomica. Vol. XV. Commentationes mechanicae ad theoriam machinarum pertinentes. Vol. primum.** Edidit J. Ackeret. Societas Scientiarum Naturalium Helveticae, Lausanne, 1957. lxi+318 pp. (3 plates).

This volume contains 9 articles, nos. 179, 833, 259, 202, 203, 222, 206, 207, 208 in Enestrom's index. There are 3 articles on Segner's hydraulic machine and one on Mour's; there are 2 articles on pumps. Of the remaining 3, one is in Latin: On the motion and reaction of water flowing in movable pipes, and 2 in French: ···machines qui sont mises en mouvement par la réaction de l'eau, and Sur le mouvement de l'eau par les tuyaux de conduite. There is a long excellent historical and analytic introduction in German by Ackeret.

Rychlík, Karel. A Cauchy manuscript in the archives of the Czechoslovakian academy of sciences. Časopis Pěst. Mat. **82** (1957), 227–228. (Czech)

Polak, L. S. William Rowan Hamilton (on the 150th anniversary of his birth). Trudy Inst. Istor. Estest. Tehn. **15** (1956), 206–276. (Russian)

A detailed consideration of Hamilton's scientific work, with a photograph.

★ **The scientific papers of Sir Geoffrey Ingram Taylor. Vol. I. Mechanics of solids. Edited by G. K. Batchelor.** Cambridge University Press, New York, 1958. x+593 pp. (18 plates) $14.50.

This volume, the first of a projected series of four, comprises Taylor's work on solid mechanics from 1917 to 1955.

It is indeed fortunate that the editor has included a series of reports written for various government agencies during the war which are here published for the first time: for instance, the classic papers "Propagation of earth waves from an explosion" (1940) and "The plastic wave in a wire extended by an impact load" (1942) which pioneered the study of plastic wave propagation.

There are 41 papers in all, in chronological order; a large proportion deals with aspects of plastic deformation, either of single crystals or, phenomenologically, of continuous bodies.

The book is very well edited and produced; it is to be hoped that supplements will have to be issued from time to time.

★ **Ore, Oystein. Niels Henrik Abel: Mathematician extraordinary.** University of Minnesota Press, Minneapolis, Minn., 1957. iv+277 pp. (14 plates) $5.75.

A non-technical biography based on much new material

obtained from private letters, official documents, and newspaper files. A great amount of detail is presented in very readable fashion.

★**Peano, Giuseppe. Opere scelte. Vol. I. Analisi matematica-Calcolo numerico.** A cura dell'Unione Matematica Italiana e col contributo del Consiglio Nazionale delle Ricerche. Edizioni Cremonese, Roma, 1957. vii+530 pp. (1 plate) 5000 Lire.

A preface by Ugo Cassini states that plans have been made to publish selected works of Peano under seven headings in three volumes. Vol. I is herewith presented with 45 articles on analysis and 5 on numerical methods. Vol. II will contain 24 articles on mathematical logic and 5 on Interlingua and algebra of grammar, while Vol. III will have 16 articles on geometry and foundations, 7 on rational mechanics and 28 miscellaneous.

This first volume also contains a photograph of Peano, a brief biography by Cassini, a list of works dedicated to Peano between 1928 and 1957, and a complete chronological list of Peano's scientific works.

★**Volterra, Vito. Opere Matematiche; Memorie e note. Vol. III; 1900–1913.** Pubblicate a cura dell'Accademia Nazionale dei Lincei cól concorso del Consiglio Nazionale delle Ricerche. Accademia Nazionale dei Lincei, Roma, 1957. 615 pp. (2 plates) 8000 Lire.

The present volume contains 41 articles. The earlier volumes are listed in MR **16**, 2 and **18**, 268.

★**Bianchi, Luigi. Opere. Vol. VII. Problemi di rotolamento.** A cura dell'Unione Matematica Italiana e col contributo del Consiglio Nazionale delle Ricerche. Edizioni Cremonese, Roma, 1957. 361 pp. 3500 Lire.

Ten articles in chronological order, written in the period 1913–1917.

★**Bianchi, Luigi. Opere. Vol. VIII. Classi speciali di superficie.** A cura dell'Unione Matematica Italiana e col contributo del Consiglio Nazionale delle Ricerche. Edizioni Cremonese, Rome, 1958. 398 pp. 3500 Lire.

This volumes includes 24 articles, with an introduction by M. Villa.

★**Ricci-Curbastro, Gregorio. Opere. Vol. II. Note e memorie: teoria dell'elasticità.** A cura dell'Unione Matematica Italiana e col contributo del Consiglio Nazionale delle Ricerche. Edizioni Cremonese, Roma, 1957. iii+586 pp. 5000 Lire.

Concluding volume of Ricci's works [for Vol. 1 see MR **17**, 932], containing papers published from 1896 to 1926 (including the monograph with Levi-Civita, "Méthodes de calcul différentiel absolu et leurs applications") and the previously unpublished "Lezioni sulla teoria matematica dell'elasticità". A table comparing notation and terminology with current usage is appended.

★**Fubini, Guido. Opere scelte. Vol. I.** A cura dell'Unione Matematica Italiana e col contributo del Consiglio Nazionale delle Ricerche. Edizioni Cremonese, Roma, 1957. 369 pp. (1 plate) 3500 Lire.

Three volumes are planned, of approximately equal length. The present volume begins with a photograph of Fubini, a preface by A. Terracini, the complete list of Fubini's publications (6 treatises and 193 articles), and a commemorative oration delivered in 1954 by B. Segre. In selecting the articles for inclusion in these three volumes, analysis has been favored and projective differential geometry relatively neglected, partly on grounds of availability of the works in question. The articles selected are printed in chronological order. The 24 articles in the present volume cover the years 1899 through 1904.

MISCELLANEOUS

★**Колмогоров, А. Н. [Kolmogorov, A. N.] О профессии математика. [Mathematics as a profession.]** 2nd ed. augmented. Gosudarstv. Izdat. "Sovetskaya Nauka", Moscow, 1954. 31 pp. 0.50 rubles.

This pamphlet is referred to in an address [Amer. Math. Monthly **64** (1957), 389–408; MR **19**, 236] by B. V. Gnedenko: "a special pamphlet by the academician A. N. Kolmogorov serves as a guide to young people in deciding on their vocation. The pamphlet has been issued in tens of thousands of copies."

There are six chapters: the nature of a mathematician's work, tests of mathematical ability, present-day machine mathematics, etc. There are also three supplements with bibliography, problems from the "Olympiads", etc. A schoolboy may consider himself gifted if at first attempt he can quickly visualize the intersection of the surface of a cube with a plane through the centre of the cube and perpendicular to one of its diagonals.

★**Perkal, Julian. Matematyka dla rolników. [Mathematics for agriculturalists.] Vol. I.** Państwowe Wydawnictwo Naukowe, Warsaw, 1958. 254 pp. (2 plates, 1 insert)

★**Градштейн, И. С. [Gradšteĭn, I. S.] Прямая и обратная теоремы. [Direct and inverse theorems.]** 2nd ed. revised. Gosudarstv. Izdat. Tehn.-Teor. Lit. Moscow -Leningrad, 1950. 80 pp. 1.70 rubles.

A slightly altered version of the first edition of 1936. The book is written for students in the Russian ten-year schools. Its purpose is to prepare for the study of higher mathematics by a careful discussion of such topics as: all, each, inverse, counter-example, generalization, set, property, necessary and sufficient, etc.

★**Richardson, M. Fundamentals of mathematics.** Revised ed. The Macmillan Company, New York, 1958. xviii+507 pp. $6.50.

The present edition, a revision of the first edition of 1941, contains much new material on social and behavioral sciences. It attempts "to provide a suitable terminal course for students of the arts and social sciences" enabling them to "learn how to use mathematicians"; that is, to "learn the nature and concepts of mathematics well enough to know what they can expect mathematicians to do for them". Representative topics are: the number system, logic of algebra, algebra of logic, calculus, probability and statistics, non-euclidean geometry, electronic computers, information theory, political structures, theory of games and strategy.

★**Fischer, Paul B. Arithmetik.** 3te Aufl. Sammlung Göschen Bd. 47. Walter de Gruyer & Co., Berlin, 1958. 152 pp. DM 2.40.

A step-by step introduction, to the complex number field, written in an elementary and interesting style with many examples. There are six sections: introductory, the

natural numbers, the integers, the rationals, the reals, the complex numbers.

Menger, Karl. Calculus 1950 — Geometry 1880. I. Scripta Math. 22 (1956), 89–96.

Menger, Karl. Calculus 1950 — Geometry 1880. II. Scripta Math. 22 (1956), 203–206 (1957).

A discussion of the parallelism that "seems to exist between calculus in 1950 (e.g., the traditional treatment of the concept "variable" leaves much to be desired) and geometry in 1880 (when Euclid's Elements was still regarded as a "matchless logical structure").

★**Steinhaus, Hugo. Sto zadań. [One hundred problems].** Państwowe Wydawnictwo Naukowe, Warsaw, 1958. 196 pp.

The problems are divided into 6 sections, with the headings: numbers, equalities and inequalities; points, polygons, circles and ellipses; space, polyhedra and spheres; problems practical and ... impractical; chess, networks, pursuit. Solutions are given in an instructive manner. There is an additional chapter on problems without solutions.

★**Günter, N. M.; und Kusmin, R. O. Aufgabensammlung zur höheren Mathematik. Bd. 2.** Hochschulbücher für Mathematik, Bd. 33. VEB Deutscher Verlag der Wissenschaften, Berlin, 1957. vi+289 pp. DM 19.60.

For the first volume see MR 19, 236. The present volume contains the remaining parts of the Russian original, namely: partial differential equations, series, approximations, functions of a complex variable, equations of mathematical physics, calculus of variations, probability.

★**Constant, F. Woodbridge. Theoretical physics: Thermodynamics, electromagnetism, waves, and particles.** Principles of Physics Series. Addison-Wesley Publishing Company, Inc., Reading, Mass., 1958. xiii+364 pp. $7.50.

This text-book for advanced undergraduates has 3 parts; thermodynamics, electromagnetism, wave particles and relativity. There are 15 chapters, on the topics: variables of state, laws of thermodynamics, thermodynamic relations, kinetic theory, classical statistical mechanics, quantum statistics, fundamental laws of electromagnetism, laws of Coulomb, Ampere and Faraday, Maxwell's postulate, electromagnetic properties of material media, waves, particles and relativity, special theory of relativity, wave mechanics. There are many problems, with answers to the odd-numbered ones.

★**Russian-English glossary of electronics and physics.** Consultants Bureau, Inc., New York, 1957. iii+354 pp. $10.00.

The present volume of this long-range undertaking (a comprehensive dictionary of contemporary Russian terminology in all fields of modern physics) has been compiled from such Russian journals as: Automation and Remote Control, Technical Physics, Radio-engineering and Electronics, Electricity, Acoustics, and Communications. It contains not only single words but also a large number of complete phrases.

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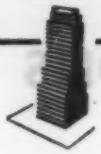

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